

MPI-PhT/95-79
LMU-TPW 95-12
August 1995

General properties of the decay amplitudes for massless particles.

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Abstract

We derive the kinematical constraints which characterize the decay of any massless particle in flat spacetime. We show that in perturbation theory the decay probabilities of photons and Yang-Mills bosons vanish to all orders; the decay probability of the graviton vanishes to one-loop order for graviton loops and to all orders for matter loops. A general power counting argument indicates in which conditions a decay of a massless particle could be possible: the lagrangian should contain a self-coupling without derivatives and with a coupling constant of positive mass dimension.

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The massless particle which we best know, the photon, is certainly stable for very long periods. The experimental evidence concerning the properties of the neutrino (admitted it is really massless) is less strong, but it is generally regarded as stable too.

Nevertheless, kinematics allows in principle the decay of a massless particle, provided the products are massless and their momenta have the same direction and versus of the initial momentum (compare Section 1). This means that the Mandelstam variables of the process vanish, so that its amplitude, regarded as a function of Mandelstam variables, must be computed in this particular limit [§]. Moreover, even if the limit of the amplitude is not zero, the phase space for the products reduces to a line in momentum space and therefore its volume tends to vanish.

An almost exact “collinearity” of the products is usually observed in the decay of any particle for which $E \gg m$. Four-momentum conservation implies (see for instance [1]) that the mass of a particle produced in an annihilation process is proportional to the sine of the angle between the momenta of the colliding particles; conversely, in the limit $m \rightarrow 0$ the decay produces collinear particles.

Limits of this kind ($m \rightarrow 0$) are common in the treatment of infrared singularities in quantum field theory ([2]; see also Sect. 5). In our case, however, the assignment of an infinitesimal mass m to the particles involved in the decay is unsuitable as a regularization technique. In fact, let us consider the decay of a massless particle of energy E into n collinear particles of the same kind, with energies E_i such that $\sum_{i=1\dots n} E_i = E$. This process is kinematically allowed (compare Section 1; n must be odd if the initial particle has nonzero helicity), but if we give the particles an infinitesimal mass, it becomes obviously impossible (suppose to observe it in the rest system of the initial particle).

We shall then work from the beginning with massless particles and introduce a different regularization, involving a weak external source J which gives the initial particle an infinitesimal additional energy (and/or momentum) ω , so that its 4-momentum is put slightly off-shell. This regularization technique proves to be quite effective, as it also allows an estimate of the decay probability by power counting.

Let us now come to the specific cases we treated. In QED it is possible to show in a general way by means of the Ward identities that the decay amplitude for $\gamma \rightarrow \gamma_1 + \dots + \gamma_n$ (n odd) is

[§]In the four-particle amplitude we mean by Mandelstam variables the usual ones, s, t, u ; for amplitudes with more external massless particles, they are taken to be all the possible scalar products between the external four-momenta.

a symmetrized sum of terms which can be factorized into a finite scalar part and a tensor part that vanishes when all the external momenta are aligned. An analogous reasoning holds for the neutrino. In both cases, it is crucial that the loop amplitudes contain in the denominator the masses of the fermions or of the vector bosons, respectively.

Another example of massless particle is the graviton. Here we do not have any experimental evidence yet. It has been suggested [4] that the non-linearity of Einstein equations could lead to a “frequency degeneration” in gravitational waves, a phenomenon which from the quantum point of view would correspond to a decay of the graviton into more gravitons of smaller energy. We were able however to prove through a generalization of the procedure applied to QED that the amplitude of this process vanishes in perturbation theory around the flat background. In this case the negative mass dimensionality of the Newton constant plays a role analogous to the fermion masses in QED. At the non perturbative level, the hypothesized existence of a small scale cosmological constant could change the situation (see below).

The case of the gluon, although physically quite academic due to the confinement, is particularly interesting because the amplitude of the decay $g \rightarrow g_1 + \dots + g_n$ (n odd) is finite for $n = 3$ and divergent for $n \geq 5$. (The Ward identities still allow a factorization of this amplitude like in QED, but the scalar parts now contain poles.) Nevertheless, the total decay probability is zero because the phase space for the products is suppressed strongly enough to compensate for the divergence in the amplitude. We thus have here a typical example of cancellation of infrared divergences in the computation of a physical quantity.

A general power counting argument indicates in which conditions a real decay of a massless particle could be possible: the lagrangian should contain a self-coupling without derivatives and with a coupling constant of positive mass dimension. This is precisely what happens in quantum gravity in the presence of a cosmological constant, and in fact it has been suggested that in this theory strong infrared effects could become relevant [5]. But one must remind that in the lagrangian the cosmological constant also multiplies a term which is quadratic in the field and thus generates an effective mass for the graviton (if $\Lambda < 0$) or an unstable theory (if $\Lambda > 0$) [6]. A possible way to elude the problem is to admit, like in lattice theory, that the effective cosmological constant vanishes on large scales but not on small scales and is negative in sign (compare Section 5). This latter approach is however out of the scope of our paper.

The structure of the article is the following. Section 1 is concerned with kinematics. In Section 1.1 we give a list of simple kinematical properties which characterize the decay of any massless particle. These properties are only due to Lorentz invariance and to the conservation

of the total four-momentum and angular momentum. In Section 1.2 we reexpress in a more manageable form the Lorentz-invariant decay measure defined on the phase space of n massless product particles, under the condition that also the initial particle is massless; specializing to the case $n = 2$ we compute explicitly the lowest order decay probability in the toy-model scalar $\lambda\phi^3$ theory. In Section 1.3 we introduce an infrared regularization which allows the computation of the decay amplitudes in the limit of vanishing Mandelstam variables. In Section 2 we give a dimensional estimate of the decay probability of the photon, the neutrino, the gluon and the graviton. After recalling in Section 3 how the exact proper vertices are connected to the complete perturbative expression for the decay amplitude, in Section 4 we use the Ward identities for QED, Yang-Mills theory (YM) and Einstein quantum gravity (QG) to give an estimate of the regularized amplitudes. In Section 5 we comment on the relation between the infrared singularities which occur in our computations and the usual infrared singularities of quantum field theory. Finally we present a few brief speculations about the possible role of a non-vanishing cosmological constant in the decay of the graviton.

1 General kinematic properties.

1.1 Consequences of Lorentz invariance.

We list here the most general properties of the decay of a massless particle. They are due only to the Lorentz invariance of the process and to the conservations of the total four-momentum and angular momentum. As we mentioned in the introduction, some of them can be proven taking the limit $m \rightarrow 0$ in the corresponding formulas for massive particles [1]. Properties 1, 2, 3, 6 can also be found in ref. [3].

Property 1. – A massless particle can only decay into massless particles. – In fact, through a suitable Lorentz boost we can make the energy of the initial state arbitrarily small. If, *per absurdum*, in the final state massive particles were present, the energy of this state would be in any reference frame equal or bigger than the sum of the masses.

Property 2. – Let us suppose that the impulse \vec{p}^0 of the initial particle is oriented in a certain direction and versus, for instance let its four-momentum have the form

$$p^0 = (E^0, E^0, 0, 0) \tag{1}$$

Then also the impulses $\vec{p}^1 \dots \vec{p}^n$ of the n product particles are oriented in the same direction

and versus; in our example we shall have (Fig. 1)

$$p^i = (E^i, E^i, 0, 0); \quad i = 1, \dots, n; \quad \sum_{i=1}^n E^i = E^0. \quad (2)$$

In an arbitrary Lorentz frame this can be rewritten as

$$p^i = \lambda_i p^0, \quad (3)$$

where $1 > \lambda_i > 0$, $i = 1, 2, \dots, n$, $\sum_i \lambda_i = 1$.

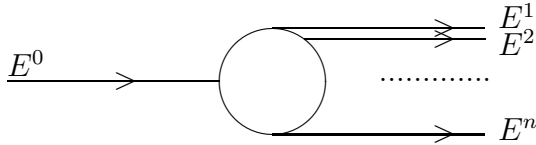


Figure 1: Collinearity property (Property 2).

– Also this property depends on the fact that through a suitable Lorentz boost along z we can make the energy of the initial state arbitrarily small; while if *per absurdum* in the final state some transversal momenta were present, their contribution to the energy would not be affected by the boost.

Property 3. – If the initial particle has helicity h and decays into n particles of the same helicity, n must be odd. – The proof follows directly from Property 2 and from the conservation of the angular momentum.

Property 4. – In the decay of a massless particle, all the scalar products $(p^i \cdot p^j)$, $i, j = 0, 1, \dots, n$ vanish. This means that the Mandelstam variables vanish. – The proof follows directly from Property 2.

Property 5. – If ε^i represents the polarization vector of the i -th particle involved in the decay, in a gauge such that $(p^i \cdot \varepsilon^i) = 0$, then we have also $(p^i \cdot \varepsilon^j) = 0$ for $i, j = 0, 1, \dots, n$. – Once more, the proof follows directly from Property 2.

Property 6. – If a massless particle decays, its lifetime τ in a reference frame where its energy is E^0 has the form

$$\tau = \xi E^0 \quad (4)$$

where ξ is a constant which depends on the dynamics of the process and has dimension $[mass]^{-2}$.

– This property holds also for massive particles, for which the constant takes the form $\xi = \tau_{rest}/m$. The proof is elementary (see for instance [3]).

1.2 The decay phase space measure $d\mu_n$.

We recall that according to quantum field theory the decay probability (per unit time) should be computed by the general formula

$$\tau^{-1} = \Gamma = \frac{1}{2E^0} \sum_{n \geq 2} \int \prod_{i=1}^n \frac{d^3 p^i}{(2\pi)^3 2E^i} \delta^4 \left(p^0 - \sum_{i=1}^n p^i \right) |T_n|^2 \quad (5)$$

where T_n is the quantum amplitude for the decay process into n product particles of momenta $\{p^i\}$. If the final particles have helicity or internal quantum symmetry numbers T_n includes the sum over these degrees of freedom.

Actually, both eq.s (4) and (5) give physically realistic predictions as far as:

- (1) the energy uncertainty ΔE of the first particle fulfils the condition $\Delta E \ll E$;
- (2)[¶] the finite energy resolution ϵ of the decay detector can be neglected. In general the detector will be unable to recognize a decay process in which one of the outcoming particles has energy E' such that $E - E' \ll \epsilon$. In order to compute the correct detection probability Γ_ϵ one should in principle subtract from formula (5) the total probability of all events of this kind. Nevertheless, for the theories considered in this paper one finds that this effect is indeed negligible (in perturbative QED, YM and QG we will find $\Gamma = 0$, whence it follows $\Gamma_\epsilon = 0$, since $\Gamma_\epsilon \leq \Gamma$; in the $\lambda\phi^3$ toy-model at order $O(\lambda^2)$ considered below one finds that $\Gamma - \Gamma_\epsilon \sim \epsilon$). We plan to devote more attention to the general issue elsewhere, by considering examples of theories for which condition (2) is not fulfilled. This requires an approach to IR divergences as in the Kinoshita-Lee-Nauenberg theorem [2].

A closer look at the measure appearing in the integrals on the RHS of formula (5) is now very useful. When all particles are massless, it is possible to express the Lorentz-invariant decay measure

$$d\mu_n = \prod_{i=1}^n \frac{d^3 p^i}{(2\pi)^3 2E^i} \delta^4 \left(p^0 - \sum_{i=1}^n p^i \right) \quad (6)$$

in the following form:

$$\begin{aligned} d\mu_n &= \frac{2\alpha_{2n-2}}{E^0} \left[\prod_{i=1}^n d^3 p^i \delta^2(\vec{p}_T^i) \theta(p_L^i) \right] \delta \left(\vec{p}^0 - \sum_{i=1}^n \vec{p}_L^i \right) \\ &\times \left[2 \left(E^0 - \sum_{i=1}^n |\vec{p}^i| \right) - \frac{1}{E^0} \left(\sum_{i=1}^n \vec{p}_T^i \right)^2 \right]^{n-2}, \end{aligned} \quad (7)$$

[¶]We thank M. Abud for drawing our attention to this point and to several further subtleties which are required by a correct physical interpretation of infrared divergences related to massless particles

where \vec{p}_L^i and \vec{p}_T^i denote the longitudinal and trasversal part of the impulse \vec{p}^i with respect to the direction and versus identified by \vec{p}^0 , θ is the step function, and the adimensional coefficients α are those which appear in the expression of $\delta^m(\vec{x})$ in polar coordinates:

$$\delta^m(\vec{x}) = \alpha_m^{-1} \delta(|\vec{x}|^2) |\vec{x}|^{2-m}. \quad (8)$$

The δ -functions occurring in formula (7) show that the support of $d\mu_n$ is concentrated around (the infinitesimal neighbourhood of) the collinearity region, which is characterized by all sets $\{p^i\}$ satisfying relation (3).

The collinearity property (3) follows from the sole condition $p^0 - \sum_{i=1}^n p^i = 0$ [imposed by the δ -function contained in formula (6)] if all p^i are null vectors. In fact, we observe that

$$\left(\sum_{i=1}^n |\vec{p}^i| \right)^2 - \left| \sum_{i=1}^n \vec{p}^i \right|^2 = \left(\sum_{i=1}^n p^i \right)^2 = (p^0)^2 = 0. \quad (9)$$

The 3-vector $\sum_{i=1}^n \vec{p}^i$ has length $\ell \leq \sum_{i=1}^n |\vec{p}^i|$, and the equality holds only if $\vec{p}^i = \lambda_i \vec{p}$, for some \vec{p} and some $\{\lambda_i\}$ all of the same sign; inserting this into the relation $p^0 - \sum_{i=1}^n p^i = 0$ we find eq. (3).

The squared amplitude $|T_n|^2$ depends only on the Lorentz invariants $(p^i \cdot p^j)$. But in the collinearity region $p^i \cdot p^j = 0$. Thus in this region a finite $|T_n|^2$ may only be a function (possibly trivial) of the invariants λ_i defined in formula (3); in this case the corresponding integral can be easily performed and gives a finite (possibly vanishing) result. In particular, when $n \geq 3$ the factor in the square bracket of formula (7) is set equals to zero by the δ -functions, and thus if the amplitude of the decay is finite, the corresponding total probability is zero. If $|T_n|^2$ diverges, we may introduce a suitable regularization in order to make the integration easier (see Section 1.3).

For a massless scalar field theory with self-coupling of the form $\lambda\phi^3$ the phase space integral (7) with $n = 2$ coincides, up to a factor λ^2 , with the probability of the decay of a particle into two particles, computed perturbatively to lowest order. This is a concrete example of computation of a finite decay probability, although with the known limitations of the $\lambda\phi^3$ theory ^{||}.

Setting $n = 2$ in (7) and performing the integral (the square amplitude does not depend on p^1 , p^2 and is equal to λ^2) we obtain

$$\int d\mu_2 = \frac{2\alpha_2}{E^0} \int dp^1 dp^2 \delta[E^0 - (p^1)^1 - (p^2)^1] \theta[(p^1)^1] \theta[(p^2)^1] = 2\alpha_2. \quad (10)$$

^{||}It is known that the action is not limited from below and that the radiative corrections do not preserve $m = 0$.

The present conclusion that Γ is finite to order λ^2 coincides with that of the dimensional analysis (considered in Section 2) applied to this case, in which the coupling constant has positive mass dimension.

1.3 Regularization through an external source.

We would like now to introduce an infrared regularization in order to allow a quick estimate of the integrals (5) in all cases (including the case in which the amplitude T_n diverges on the collinearity region).

Obtaining such a regularization is not trivial. The most common infrared regularization technique, which consists in giving the soft particles a small mass μ which eventually goes to zero, does not work in the present case, because the (regularized) process in which one particle of mass μ decays into more particles of the same mass has obviously zero probability.

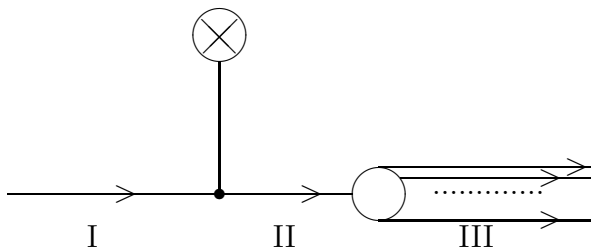


Figure 2: Factorization of the decay amplitude.

Instead, a better approach is to put external momenta slightly off-shell in a way controlled by an infinitesimal parameter ω .

Let us suppose (Fig. 2) that a very weak external source J gives the decaying particle (state I) an infinitesimal additional energy ω . The exact nature of the source and of the particle which carries the energy ω are not essential. For instance, if J represents a classical field, the energy can be carried by an on-shell boson with four-momentum $(\omega, 0, 0, \omega)$; by absorbing the boson, the initial particle gains a small transversal impulse (state II) too. Alternatively, the energy ω could be carried by an off-shell boson produced in J through an annihilation process, with four-momentum $(\omega, 0, 0, 0)$; by absorbing the boson, the initial particle gets off shell too. More generally, we will assume that after the interaction the four-momentum of the initial particle will have the form

$$p^\mu = \hat{p}^\mu + \omega b^\mu, \quad (11)$$

where b^0 is adimensional and $(\hat{p}^0)^2 = 0$. It is not necessary to make any special assumption on the four-vector b^0 at this stage (in Sections 3, 4 we will prefer to specialize the discussion by assuming that $b^0 \cdot p^0 = 0$, and other similar conditions for the p^i 's).

At this point the decay takes place; the products (state III) have now a small transversal impulse of order ω and the Mandelstam variables ($p^i \cdot p^j$) are of order ω (at least). The partial decay probability into n product particles is written as a sum over intermediate states (compare (eq. 5))

$$\Gamma_n = \lim_{\omega \rightarrow 0} \frac{1}{2E_\omega^0} \int \prod_{i=1}^n \frac{d^3 p^i}{(2\pi)^3 2E^i} \delta^4 \left(p_\omega^{II} - \sum_{i=1}^n p^i \right) |\langle II_\omega | \mathcal{T} | III \rangle|^2 \quad (12)$$

where \mathcal{T} is the appropriate evolution operator. When $\omega \rightarrow 0$, the factor $1/E_\omega^0$ tends to $1/E^0$, which is the dependence that we expect on the basis of Lorentz invariance (compare Property 6). Thus in this limit the integral I_n appearing in the preceding formula does not depend on E . Summing up we obtain

$$\Gamma_n = \frac{1}{2E^0} \lim_{\omega \rightarrow 0} I_n(\omega); \quad (13)$$

the only the only massive parameters on which $I_n(\omega)$ depends are ω and the massive parameters possibly present in the theory that we are considering. This allows in most cases to estimate dimensionally whether Γ_n is finite, vanishes or diverges in the limit $\omega \rightarrow 0$ [note that the mass dimensions of I_n , $|\langle II_\omega | \mathcal{T} | III \rangle|^2$ and of $d\mu_n$ are respectively equal to $2, 2(3-n), 2n-4$]. We shall give some examples of this in the next Section.

2 Power counting.

In several cases the integral I_n can be estimated by simple arguments (often dimensional considerations alone are enough).

For instance, in QED the four-photons amplitude is given to lowest order by the four fermions loop (fig. 3a). It is easy to realize that the loop integral gives a 4-th degree homogeneous polynomial in the dimensionless variables $\frac{p^i}{m_f}$ [7], where m_f is the mass of the fermion. The integral I_3 will therefore be proportional to

$$I_3 \sim \alpha^4 \left(\frac{\omega}{m_f} \right)^8 \omega^2 \quad (14)$$

where α is the fine structure constant. All behaves as though

$$T_3 \sim \alpha^2 \left(\frac{\omega}{m_f} \right)^4. \quad (15)$$

To be precise, the behaviour $T_3 \sim \omega^4$ holds only for some specific choices of the "slightly off-shell" external momenta p^i , whereas in any case $T_3 = O(\omega^2)$ at least; the integration transforms the remaining dependence of T_3 on p^i , if any, into an additional ω^2 factor.

The above result can be generalized to the n -fermions loop: the key point is that the fermionic propagators of the loop produce masses in the denominator. The case of the neutrino is analogous: the masses of Z^0 or W^\pm appear at the denominator in the amplitude. In both cases, since the amplitude is proportional to a positive power of the regularizator ω , it vanishes in the infrared limit due to (12).



Figure 3: (a) Fermions square loop. (b) Gravitons or gluons loop.

In the case of pure quantum gravity we have tree and one-loop graviton diagrams with k external legs (fig. 3b). Explicit expressions for the $k = 4$ amplitudes have been given by [8, 9]. In any case, these amplitudes contain positive powers of the constant $\kappa = \sqrt{16\pi G}$ and then, like in QED, they behave always like a positive power of ω and cause the decay probability to vanish.

In the case of QCD the amplitudes do not contain dimensional constants. We expect that the decay amplitude of the gluon into three gluons, being adimensional, tends to a constant when $\omega \rightarrow 0$, and this is in fact what happens [9]. The decay amplitudes of a gluon into 5, 7 ... gluons have mass dimensions -2, -4 ... respectively, so they diverge when $\omega \rightarrow 0$; but this divergence is compensated in the phase space integral by a bigger positive power of ω in such a way that the probability behaves like $\omega^2/E^{(0)}$ and thus vanishes in the limit.

We are not going to apply this power counting argument to all possible theories and couplings, since it is in each case quite immediate. As a last example, we may wonder whether a photon can in principle decay due to the gravitational interaction, through diagrams with external photons and one loop of gravitons. Since the coupling constant κ has mass dimension -1, while the fine structure constant α is adimensional and there are no masses involved, we

conclude once more that the amplitude of the process vanishes in the infrared limit.

It is clear from the discussion above that a Γ_n different from zero can be only obtained when the square amplitude is proportional to a sufficiently high negative power of ω . Since in perturbation theory the coupling constants always appear in the numerator, this means that the amplitude must contain a coupling constant with positive mass dimension. We shall return on this point in the conclusions.

3 Diagrammatics: ω -dependence of the decay amplitudes.

The dimensional arguments of the previous section determine the ω -dependence of the decay probability only for the pure gauge theories (YM, QG), where the only parameter in the action is the coupling constant. If additional dimensionful parameters appear in the action (as it happens for instance when the gauge field is coupled to some massive field) the previous arguments, as we have seen in the QED example, must be completed by some additional information. In general, a more explicit analysis of the perturbative expansion and use of Feynman diagrams is therefore needed in order to estimate the total decay probability. In this and in the following section we carry it out in such a way to determine not only the ω -dependence of the total decay probability, but also of the decay amplitudes (i.e., of the probabilities of the single decay channels). The general results for the former will be essentially the same as those found by the dimensional arguments in section 2. Thus, we conclude that the decay probability of the gauge bosons of QED, YM, QG vanish.

Before starting, let us define a “decay configuration” as follows: it is a pair of $(n + 1)$ four-momenta and $(n + 1)$ polarization vectors $(p^i, \varepsilon^i)_{i=0,1,\dots,n}$ satisfying the properties $(p_i)^2 = 0$, $\sum_{i=0}^n p^i = 0$, $(\varepsilon^i \cdot p^i) = 0$, $p_0^0 > 0$, $p_0^l < 0$ for $l = 1, \dots, n$. We thus agree that the signs of the four-momenta of the outgoing particles are reversed. As we have seen, for particles with non-zero helicity n must be odd.

We will start the analysis of the perturbative expansion from the tree level: a sum of truncated connected tree-diagrams with $(n + 1)$ external lines will give the lowest order (in \hbar) contribution to the decay amplitude of 1 gauge boson in n gauge bosons. Higher order corrections will involve truncated connected diagrams with one or more loops. To formally compute the “exact” decay amplitude one has to replace in each tree diagram every boson propagator with the corresponding exact boson propagator, and each m -boson vertex with the

corresponding m -boson proper vertex (i.e. one-particle-irreducible Green function)**. In order to get the \hbar^r -order approximation of the decay amplitude, one simply has to retain the terms of order $\leq r$ in this formal “exact” expression. As we will see, the Ward identities imply that when approaching a decay configuration: (1) in QED the decay amplitude of a process with m external photons vanishes; (2) in QG the decay amplitude of a process with m external gravitons or photons vanishes; (3) the decay amplitudes of processes with external Y.M. bosons may be finite or diverge, but in such a way that the corresponding decay probabilities vanish.

3.1 Tree level

Let us start from the Feynman vertices with m gauge massless bosons ($m \geq 3$) [see the actions (25)]: we draw them in fig. (4). The diagrams are to be understood as truncated in the external lines. In QED there is no m -photon vertex. In YM there are only two m -gluon vertices (for $m = 3, 4$). In pure QG there is one m -graviton vertex for every $m \geq 3$; if coupling of gravity with the electromagnetic or the Yang-Mills fields is considered, then there are also vertices with k spin-1 bosons (photons or gluons) and r gravitons, for $k = 2, 3, 4$ and $r \geq 1$. In the figures, a wavy line in the QG case will denote either a graviton or another gauge boson (a photon or a gluon).

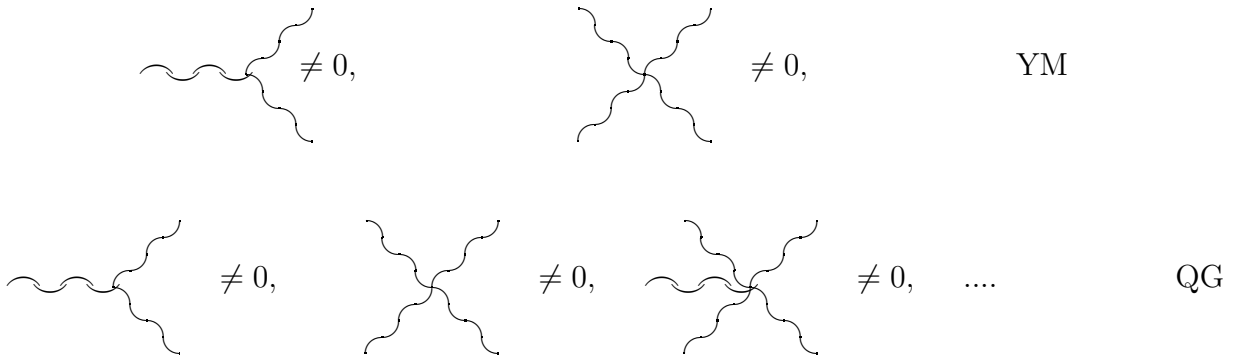


Figure 4: Feynman vertices

At the tree level, the decay amplitudes $T_2^{tree}, T_3^{tree}, T_4^{tree}, T_5^{tree}, \dots$ of YM, QG are the sum of the diagrams in fig. (5).

Tree diagrams involving ghost lines do not contribute to T_n^{tree} . In fact, even though ghosts are massless, diagrams with external ghosts are zero when multiplied by physical polarization

**In principle, propagators and proper vertices could be computed even in two different gauges, in order to simplify calculations, see Ref. [10]

$$\begin{aligned}
T_2^{tree} &= \text{[Diagram: a wavy line on the left connected to a vertex with two outgoing wavy lines]} \\
T_3^{tree} &= \text{[Diagram: a vertex with three outgoing wavy lines]} + \text{[Diagram: a wavy line on the left connected to a vertex with two outgoing wavy lines]} \\
T_4^{tree} &= \text{[Diagram: a vertex with four outgoing wavy lines]} \text{ (QG)} + \text{[Diagram: a wavy line on the left connected to a vertex with two outgoing wavy lines]} \\
T_5^{tree} &= \text{[Diagram: a vertex with five outgoing wavy lines]} \text{ (QG)} + \text{[Diagram: a wavy line on the left connected to a vertex with two outgoing wavy lines]} \text{ (QG)} + \text{[Diagram: a wavy line on the left connected to a vertex with two outgoing wavy lines]} + \\
&\quad \text{[Diagram: a vertex with three outgoing wavy lines]} + \text{[Diagram: a wavy line on the left connected to a vertex with two outgoing wavy lines]}
\end{aligned}$$

Figure 5: Tree level amplitudes: (QG) means that the diagram in T_5^{tree} is present only in QG.

vectors, and diagrams with internal ghost lines (propagators) have necessarily also external ghost lines, by ghost number conservation. One can easily verify that in QG the decay amplitudes with only m external gravitons or photons vanish ($T_n^{tree} = 0$) in any decay configuration, because each vertex is quadratic in the momenta k^i , implying an overall $(k)^2$ dependence of each separate diagram in fig. (5); when contracted with the external polarization vectors, this will give zero, since in the decay configuration all 4-momenta are null vectors proportional to each other.

3.2 Higher orders

To formally compute the “exact” decay amplitude one has to replace in each tree diagram every boson propagator with the corresponding exact boson propagator, and each m -boson vertex with the corresponding m -boson proper vertex (i.e. one-particle-irreducible Green function), as

depicted in fig. (6); there we have symbolized each proper vertex by a blob. Diagrams involving ghost lines can be excluded for the same reasons as before.

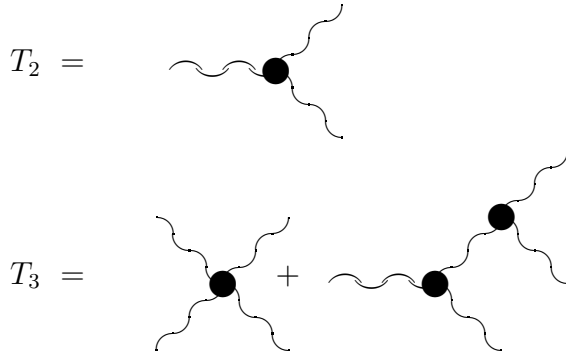


Figure 6: Exact amplitudes

Using Property 2 it is easy to verify that if the external momenta are slightly off-shell, the momenta carried by the propagators in figg. (5), (6) also are, and the scalar products of all momenta are of order ω ; ω is the infrared regulator (with dimension of a mass) introduced in section 1. The exact propagators for massless particles in the infrared limit have to behave as the naive ones, i.e. they are of order ω^{-2} .

Let E_γ, E_y, E_g and I_γ, I_y, I_g denote respectively the number of external and internal photon, YM boson, graviton lines coming out of one of the diagrams in fig. (6). Let m_γ^v, m_y^v, m_g^v denote the numbers of photons, YM bosons, gravitons coming out from the v^{th} proper vertex Γ^v appearing in the same diagram. Clearly,

$$\begin{aligned} E_\gamma &= \sum_v m_\gamma^v - 2I_\gamma \\ E_y &= \sum_v m_y^v - 2I_y \\ E_g &= \sum_v m_g^v - 2I_g. \end{aligned} \tag{16}$$

Moreover,

$$N_p - I_p \geq \theta(E_p) \tag{17}$$

where $\theta(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$ and N_p denotes the number of proper vertices in the diagram where at least one particle p (p being a YM boson and/or a graviton) comes out; this inequality follow from the fact that $N_p = 0$ if and only if $E_p = 0 = I_p$.

The results of the next section (Property 10) can be summarized as follows, that

$$\Gamma^v = o(\omega^{m_\gamma^v + 4\theta(m_y^v) - m_y^v + 2\theta(m_g^v)\delta_0^{m_y^v}}) \tag{18}$$

where in our notation $o(\omega^p)$ will denote an infinitesimal or an infinite of *at least* order p in ω , namely $\lim_{\omega \rightarrow 0} [o(\omega^p)\omega^{-p}]$ is zero or finite. The overall ω -dependence of the diagram contribution $D(\omega)$ will be the product of the dependences of each vertex and each propagator:

$$D(\omega) = \left[\prod_v o(\omega^{m_\gamma^v + 4\theta(m_y^v) - m_y^v + 2\theta(m_g^v)\delta_0^{m_y^v}}) \right] \omega^{-2(I_\gamma + I_y + I_g)} \quad (19)$$

Using equations (16), the latter becomes

$$D(\omega) = o(\omega^{E_\gamma - E_y + 4(N_y - I_y) + 2(N'_g - I_g)}), \quad (20)$$

where N'_g denotes the number of proper vertices in the diagram where at least one graviton and no YM boson come out. To estimate $4(N_y - I_y) + 2(N'_g - I_g)$ let us distinguish two cases. If $E_y = 0$, then by colour conservation $m_y^v = 0$ for all vertices in the diagram, implying $N'_g = N_g$; using formulae (17) for $p = y$ and $p = g$, we find $4(N_y - I_y) + 2(N'_g - I_g) \geq 4\theta(E_y) + 2\theta(E_g)$. If $E_y > 0$, noting that $(N_y + N'_g) = N_p$, $I_y + I_g = I_p$, where now p denotes *either y or g*, and using formulae (17), we find $4(N_y - I_y) + 2(N'_g - I_g) \geq 2\theta(E_y) + 2\theta(E_p) = 4\theta(E_y)$. Summing up, $4(N_y - I_y) + 2(N'_g - I_g) \geq 4\theta(E_y) + 2\theta(E_g)\delta_0^{E_y}$. This expression depends only on the numbers of external bosons of the process, not on the particular diagram we are considering, therefore we find the following

Property 7. – The amplitude T of a decay process with E_γ external photons, E_y external YM boson and E_g gravitons satisfies the condition:

$$T = o(\omega^{E_\gamma - E_y + 4\theta(E_y) + 2\theta(E_g)\delta_0^{E_y}}). \quad (21)$$

This formula is valid at any loop order in all particles different from the gravitons and at least at one loop order in the gravitons, because the matter action with a background metric is multiplicatively renormalizable [13], whereas at first order in the graviton loops pure QG is finite on-shell.

Note that the RHS of formula (21): 1) is independent of the number of external gravitons, provided $E_y > 0$; 2) vanishes if $E_y = 0$.

4 Ward identities

In QED the proper n -photon vertices $\Gamma_n^{\mu_1 \dots \mu_n}(p^1, \dots, p^n)$ satisfy the Ward identity

$$p_{\mu_1} \Gamma_n^{\mu_1 \mu_2 \dots \mu_n}(p^1, p^2, \dots, p^n) \varepsilon_{\mu_2}(p^2) \dots \varepsilon_{\mu_n}(p^n) = 0, \quad (22)$$

where p^i is the momentum of the i -th photon and $\varepsilon_{\mu_i}(p^i)$ the corresponding polarization vector; this transversality condition amounts to the gauge invariance of any physical process involving n (incoming or outgoing) photons.

In this section we first derive the identity above and its analogues for general Yang-Mills (YM) and Einstein (with $\Lambda = 0$) Quantum Gravity (QG) theories in the momentum configuration of decay processes (compare with Property 2). Then we use them and a continuity argument to show that the proper vertex for any decay process with fixed external momenta vanishes in QED and QG, whereas it is finite in YM. The Ward identities are derived formally by using naive functional integration considerations based only on the gauge invariance of the classical action (not on its explicit form). In the case of QED, YM, their validity extends to the true (i.e. renormalized) theories at any order in the loops because renormalization preserves Ward identities. In the case of QG, their validity is guaranteed at any loop order in the matter fields and at least at one loop order in the gravitons, because the matter action with a background metric is multiplicatively renormalizable [13], whereas at first order in the graviton loops pure QG is finite on-shell.

We start by fixing the notation. Let $\mathcal{S}_{inv}(\phi)$ denote the (local) action depending on the classical fields $\{\phi_I\}$ and $R_\alpha^I(\phi)$ corresponding (local) gauge generators:

$$\delta_\xi \mathcal{S}_{inv} = \frac{\delta \mathcal{S}_{inv}}{\delta \phi_I} \delta_\xi \phi^I = 0, \quad (23)$$

We employ a condensed notation in which a capital index I is a collective index; it represents both discrete indices and a continuous space-time variables x . A repeated index implies summation over discrete indices and integration over x . Explicitly, in the case of QED, YM, QG the fields ϕ_I include

$$\phi_I := \begin{cases} A_\mu(x), \psi(x), \bar{\psi}(x) \text{ and/or } \varphi(x), \bar{\varphi}(x) \text{ in QED;} \\ A_\mu^a(x), + \text{ possibly } \psi^i(x), \bar{\psi}^i(x) \text{ and/or } \varphi^i(x), \bar{\varphi}^i(x) \text{ in YM;} \\ h_{\mu\nu}(x) + \text{ possibly any } \phi_I \text{ considered in the two previous cases in QG;} \end{cases} \quad (24)$$

$x \in M^4$ denotes the point in Minkowski spacetime, $A_\mu(x), A_\mu^a(x)$ the gauge potentials corresponding respectively to a $U(1)$ and a semisimple group G , $\psi(x), \bar{\psi}(x)$ (resp. $\varphi(x), \bar{\varphi}(x)$) spinors (complex scalars), $\psi^i(x), \bar{\psi}^i(x)$ (resp. $\varphi^i(x), \bar{\varphi}^i(x)$) spinors (complex scalars) making up a finite multiplet belonging to some finite representation $Rep(\mathcal{L}ie(G))$ (in the latter case $(T^a)_j^i$ will denote the matrix representation of the hermitean Lie algebra generators corresponding to A_μ^a), $h_{\mu\nu}(x)$ is the graviton field, $\eta_{\mu\nu}$ denotes the Minkowski metric tensor (which plays the role of background metric) in cartesian coordinates, and $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ is the the metric tensor.

The invariant actions \mathcal{S}_{inv} read

$$\mathcal{S}_{inv} = \begin{cases} -\frac{1}{4} \int_{M^4} d^4x (F^{\mu\nu} F_{\mu\nu}) + \mathcal{S}_{mat} & \text{in QED;} \\ -\frac{1}{4} \int_{M^4} d^4x (F^a{}^{\mu\nu} F_{\mu\nu}^a) + \mathcal{S}_{mat} & \text{in YM;} \\ \int_{M^4} d^4x g^{\frac{1}{2}} (\lambda - \frac{1}{16\pi G} R) + \mathcal{S}_{mat} & \text{in QG,} \end{cases} \quad (25)$$

where $F_{\mu\nu}, F_{\mu\nu}^a$ is the field strength in QED, YM respectively, R is the Ricci scalar of the metric $g_{\mu\nu}$, $g := -\det[g_{\mu\nu}]$, f^{abc} are the structure constants of $\mathcal{L}ie(G)$ and e the coupling constant. \mathcal{S}_{mat} is the action of the matter minimally coupled to the gauge potential ^{††}.

$A_\mu, A_\mu^a, h_{\mu\nu}$ are respectively the gauge potentials for QED, YM, QG, with gauge transformations

$$\delta_\xi A_\mu = \partial_\mu \xi \quad \text{in QED;} \quad (26)$$

$$\delta_\xi A_\mu^a = (D_\mu \xi)^a := \partial_\mu \xi^a + e f^{abc} A_\mu^b \xi^c \quad \text{in YM;} \quad (27)$$

$$\delta_\xi g_{\mu\nu} = g_{\nu\rho} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho + \xi^\rho \partial_\rho g_{\mu\nu},$$

$$\delta_\xi A_\mu^a = A_\rho^a \partial_\mu \xi^\rho + \xi^\rho \partial_\rho A_\mu^a \quad \text{in QG.} \quad (28)$$

We omit for the sake of brevity the well-known gauge transformations of the other fields.

The quantization of the theory (in a perturbative setting) is performed in the BRST formalism [12, 11]: the set of fields $\{\phi_I\}$ is enlarged to a set $\{\Phi_A\}$ by the introduction of ghosts, antighosts and Stueckelberg fields, and we associate to the action \mathcal{S}_{inv} a gauge-fixed action S_Ψ depending on the gauge-fixing functional Ψ . Index A , like I , represents both discrete indices and the continuous space-time variables x . Let $S_{GF} := S_\Psi(\Phi) - \mathcal{S}_{inv}(\phi)$; in QED and YM, S_{GF} can be constructed as $S_{GF} = s\Psi$, where s denotes the BRST transformation associated to the gauge transformations (26) - (28).

The generating functional $Z(J)$ (depending on the external sources J) for the Green functions of the theory is defined by

$$Z(J) := \int \mathcal{D}\Phi e^{\frac{i}{\hbar} [S_\Psi(\Phi) + J^A \Phi_A]}, \quad (29)$$

where $\mathcal{D}\Phi$ is a gauge invariant functional measure, J^A transforms under diffeomorphisms as the appropriate tensor density.

^{††}Strictly speaking, in the case of QG an action \mathcal{S}_{mat} containing a spinor contribution requires the introduction of vierbeins as dynamical variables instead of the metric. However, the considerations of this section hold also in that case, since they are based on the gauge transformations (28) of the metric, which can be obtained from the gauge transformations of the vierbeins.

By performing a gauge \ddagger transformation $\phi \rightarrow \phi + \delta_\xi \phi$ of the dummy integration variables ϕ in the RHS of eq. (29) the integral $Z(J)$ remains the same (the Jacobian is 1), implying the Ward identities

$$0 = \delta_\xi Z(J) = \frac{i}{\hbar} \int \mathcal{D}\Phi [J^A \delta_\xi \Phi_A + \delta_\xi S_{GF}] e^{\frac{i}{\hbar}[S_\Psi(\Phi) + J^A \Phi_A]}, \quad (30)$$

or, in terms of the generating functional $W(J) := \frac{\hbar}{i} \ln[Z(J)]$ of the connected Green functions,

$$0 = \left[J^A \delta_\xi \Phi_A + \delta_\xi S_{GF} \right] \Big|_{\Phi_A \rightarrow \frac{\delta}{\delta J^A}} W(J) + \text{disconnected terms}. \quad (31)$$

The disconnected terms are absent when evaluating the Green function on any decay process, since in this case only *one* initial particle is present. Therefore, as far as we are concerned,

$$0 = \left[J^A \delta_\xi \Phi_A + \delta_\xi S_{GF} \right] \Big|_{\Phi_A \rightarrow \frac{\delta}{\delta J^A}} W(J). \quad (32)$$

In order to obtain the Ward identities for the proper vertex functions we introduce the usual Legendre transform $\Gamma(\tilde{\Phi}) := [W(J) - J^A \Phi_A] \Big|_{J=J(\tilde{\Phi})}$, where the function $J = J(\tilde{\Phi})$ is obtained by inverting the relations $\tilde{\Phi}_A = \frac{\delta W}{\delta J^A}$; the new independent variables are the ‘‘classical fields’’ $\tilde{\Phi}$. Consequently $J^A(\tilde{\Phi}) = -\frac{\delta \Gamma}{\delta \tilde{\Phi}_A}$.

From identity (32) we draw the following Ward identities for the generating functional of proper vertices Γ

$$0 = \left[\frac{\delta \Gamma}{\delta \tilde{\Phi}_A} \cdot \delta_\xi \tilde{\Phi}_A + \delta_\xi S_{GF}(\tilde{\Phi}) \right]. \quad (33)$$

Actually, we are interested in the Ward identities for the proper vertices having only physical gauge bosons as external (incoming or outgoing) particles. The physicality condition is best imposed in momentum space. The proper vertex $\Gamma_n^{12\dots n}(x^1, x^2, \dots, x^n)$ with n external gauge bosons $b_i(x^i)$ (in configuration space) is obtained from Γ through differentiation,

$$\Gamma_n^{12\dots n}(x^1, x^2, \dots, x^n) = \frac{\delta^n \Gamma}{\delta b_1(x^1) \dots \delta b_n(x^n)} \Big|_{\tilde{\Phi}=0}, \quad (34)$$

where we have introduced the short-hand notation

$$i \rightarrow \begin{cases} \mu_i \\ (\mu_i, a_i) \\ \mu_i \nu_i \text{ or } (\mu_i, a_i) \end{cases} \quad b_i \rightarrow \begin{cases} \tilde{A}_{\mu_i} & \text{in QED} \\ \tilde{A}_{\mu_i}^a & \text{in YM} \\ \tilde{h}_{\mu_i \nu_i} \text{ or } A_{\mu_i}^a & \text{in QG;} \end{cases} \quad i = 1, 2, \dots, n. \quad (35)$$

The RHS has automatically the required boson symmetry in the identical particles, e.g. if all the b_i 's are the same type of fields

$$\Gamma_n^{i_1 i_2 \dots i_n}(x^{i_1}, x^{i_2}, \dots, x^{i_n}) = \Gamma_n^{12\dots n}(x^1, x^2, \dots, x^n), \quad (36)$$

\ddagger Alternatively, one could perform a BRST transformation; the resulting Ward identities would be the same.

where (i_1, i_2, \dots, i_n) is a permutation of $(1, 2, \dots, n)$. On account of the translation invariance $\Gamma_n^{1\dots n}(x^1, \dots, x^n) = \Gamma_n^{1\dots n}(x^1 + a, \dots, x^n + a)$, its multiple Fourier transform can be written as $\Gamma_n^{1\dots n}(p^1, \dots, p^n) \delta^4(\sum_{i=1}^n p^i)$; it contains a Dirac- δ implementing the total momentum conservation. Here and below our conventions for the Fourier transform will be $f(p) := \int \frac{d^4x}{(2\pi)^4} e^{-ip \cdot x} f(x)$, $f(x) = \int d^4p e^{ip \cdot x} f(p)$. As a consequence of the general relation

$$\int \frac{d^4x}{(2\pi)^4} e^{-ip \cdot x} \frac{\delta F}{\delta \phi(x)} = (2\pi)^{-4} \frac{\delta F}{\delta \phi(-p)} \quad (37)$$

one finds

$$\delta^4 \left(\sum_{i=1}^n p^i \right) \Gamma_n^{12\dots n}(p^1, p^2, \dots, p^n) = (2\pi)^{-4n} \frac{\delta^n \Gamma}{\delta b_1(-p^1) \dots \delta b_n(-p^n)} \Big|_{\tilde{\Phi}=0}. \quad (38)$$

Differentiating relation (33) with respect to $b_1(-p^1), \dots, b_n(-p^n)$ and setting thereafter $\tilde{\Phi} = 0$, we obtain

$$\begin{aligned} 0 = & \int d^4q \left[(2\pi)^4 \delta^4 \left(q + \sum_{j=1}^n p^j \right) \Gamma_{n+1}^{01\dots n}(q, p^1, \dots, p^n) \delta_\xi b_0(q) \right. \\ & + \sum_{h=1}^n \delta^4 \left(q + \sum_{j=1, j \neq h}^n p^j \right) \Gamma_n^{01\dots, h-1, h+1, \dots, n}(q, p^1, \dots, p^{h-1}, p^{h+1}, \dots, p^n) \frac{\delta(\delta_\xi b_0(q))}{\delta b_h(-p^h)} \\ & \left. + \frac{\delta^n \delta_\xi S_{GF}(\tilde{\Phi})}{\delta b_1(-p^1) \dots \delta b_n(-p^n)} \right] \Big|_{\tilde{\Phi}=0}. \quad (39) \end{aligned}$$

In fact, only the terms with $\tilde{\Phi}_A = b$ in the first term in eq. (33) contribute to eq (39), since when $\tilde{\Phi}_A \neq b$ then $\frac{\delta^m(\delta_\xi \tilde{\Phi}_A)}{\delta b_1 \dots \delta b_m} \Big|_{\tilde{\Phi}=0} = 0$ (indeed, for any $\tilde{\Phi}_A$ $\delta_\xi \tilde{\Phi}_A$ is of degree ≥ 1 in $\tilde{\Phi}_A$).

To get identities involving proper vertices with *physical* external bosons we will have to contract their Lorentz indices with the ones of transverse polarization tensors/vectors (we will choose them with well-defined helicity) $e^1(p^1) \dots e^n(p^n)$, where

$$e(p) = e^\pm(p) := \begin{cases} \varepsilon_\mu^\pm(p) & \text{when } b = \tilde{A}_\mu, \tilde{A}_\mu^a \\ (\varepsilon_\mu^\pm(p) \varepsilon_\nu^\pm(p)) & \text{when } b = \tilde{h}_{\mu\nu}, \end{cases} \quad \text{with } \varepsilon_\mu^\pm(p) p^\mu = 0. \quad (40)$$

Now it is easy to realize that in all cases the following property holds:

$$\frac{\delta^n \delta_\xi S_{GF}(\tilde{\Phi})}{\delta b_1(p^1) \dots \delta b_n(p^n)} \Big|_{\tilde{\Phi}=0} e_1^1(-p^1) \dots e_n^n(-p^n) = 0; \quad (41)$$

where contraction of the Lorentz indices hidden in the symbols $1, \dots, n$ and e^1, \dots, e^n is understood. In fact, the terms of non-zero degree in the ghosts contained in $\delta_\xi S_{GF}$ vanish after setting $\tilde{\Phi}' = 0$; the other terms depend on the longitudinal modes of the bosons, and vanish after contraction with the polarization vectors/tensors. We prove explicitly this statement in the appendix, for the Feynman (harmonic) gauge fixings.

Introducing the notation

$$\Gamma_n^{1\dots e^i\dots n} := \Gamma_n^{1\dots i\dots n} \cdot e^i, \quad (42)$$

where again contraction of the Lorentz indices hidden in the symbols i and e^i is understood, the Ward identities (43) will therefore reduce to

$$\begin{aligned} 0 = & \int d^4q \left[(2\pi)^4 \delta^4 \left(q + \sum_{j=1}^l p^j \right) \Gamma_{n+1}^{0e^1\dots e^n} (q, p^1, \dots, p^n) \delta_\xi b_0(q) \right. \\ & \left. + \sum_{h=1}^n \delta^4 \left(q + \sum_{j=1, j \neq h}^n p^j \right) \Gamma_n^{0e^1\dots e^{h-1}, e^{h+1}, \dots, e^n} (q, p^1, \dots, p^{h-1}, p^{h+1}, \dots, p^n) \frac{\delta(\delta_\xi b_0(q))}{\delta b_h(-p^h)} e^h \right] \Big|_{\tilde{\Phi}=0} \end{aligned} \quad (43)$$

The identity above is one essential ingredient that we need in order to prove the main property of this section. In order to formulate this property, we need now a notion of “vicinity” of a “decay configuration” parametrized by one regularization parameter ω . Therefore, we introduce some useful definitions.

A configuration ω -converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0, \dots, n}$ ($\omega \geq 0$) is a one-parameter family $(k^i(\omega), \varepsilon^i(\omega))_{i=0, \dots, n}$ such that $\varepsilon^i(\omega) \cdot k^i(\omega) = 0$, $k^i(\omega) - \hat{k}^i = o(\omega)$, $\varepsilon^i(\omega) - \hat{\varepsilon}^i = o(\omega)$, $k^i \cdot k^{i'} = o(\omega^2) \forall i, i' = 0, 1, \dots, n$. Examples of these families will be given in formulae (73), (80).

It is easy to show that in the mentioned hypotheses the 3-momenta are in general no more collinear, but form angles $\lesssim \omega$; consequently,

$$\varepsilon^i(k^i) \cdot \varepsilon^j(k^j) = \begin{cases} \text{either } o(1) \\ \text{or } o(\omega) \end{cases} \quad \varepsilon^i(k^i) \cdot k^j = o(\omega). \quad (44)$$

We are now able to prove the following fundamental property of the vertices, which is the main result of this Section and adds to the kinematical properties of Section 1:

Property 10. – On any configuration $(k^i(\omega), \varepsilon^i(\omega))_{i=0, \dots, n}$ ω -converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0, \dots, n}$

$$\Gamma_{n+1}^{e^0\dots e^n} (k^0, \dots, k^n) = o(\omega^{n+1}) \quad \text{in QED}; \quad (45)$$

$$\Gamma_{n+1}^{e^0 a_0 \dots e^n a_n} (k^0, \dots, k^n) = o(\omega^{4-n-1}) \quad \text{in YM}; \quad (46)$$

$$\Gamma_{n+1}^{e^0 \dots e^n} (k^0, \dots, k^n) = o(\omega^{m_\gamma + \theta(m_y)(4-m_y) + 2\theta(m_g)\delta_0^{m_y}}) \quad \text{in QG}. \quad (47)$$

where in the third equation m_γ, m_y, m_g denote the number of external photons, YM bosons and gravitons respectively ($m_\gamma + m_y + m_g = n + 1$), and $\theta(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$.

Proof.

The claim is evidently true when $n = 0$. In fact, $\Gamma_1^{\mu_0} \propto (k^0)^{\mu_0}$ in QED, YM, but this vanishes since momentum conservation imposes the condition $k^0 = 0$; in QG still it could be $\Gamma_1^{\mu_0\nu_0} = \text{const} \times \eta^{\mu_0\nu_0}$, but this vanishes after contraction with $e^{\mu_0\nu_0}$ (which is a traceless tensor).

The rest of the proof is by induction and divided in three parts. Let us assume that the claim is true when $n = m - 1$. We will prove that it is true when $n = m$. For the sake of simplicity, we explicitly prove the claim (47), which is the most general possible, in the simpler case $m_\gamma = 0 = m_y$,

$$\Gamma_{n+1}^{e^0 \dots e^n}(k^0, \dots, k^n) = o(\omega^2) \quad \text{in QG;} \quad (48)$$

at the end of this section we will briefly sketch how the proof goes in the general case.

Part 1 Here we prove the equations

$$\Gamma_{n+1}^{e^0 \dots e^{i-1}, \mu_i, e^{i+1} \dots e^n}(k^0, \dots, k^n) k_{\mu_i}^i = 0 \quad \text{in QED;} \quad (49)$$

$$\Gamma_{n+1}^{e^0 a_0 \dots e^{i-1} a_{i-1}, \mu_i a_i, e^{i+1} a_{i+1} \dots e^n a_n}(k^0, \dots, k^n) k_{\mu_i}^i = o(\omega^{4-n}) \quad \text{in YM;} \quad (50)$$

$$\Gamma_{n+1}^{e^0 \dots e^{i-1}, \mu_i \nu_i, e^{i+1} \dots e^n}(k^0, \dots, k^n) k_{\mu_i}^i = o(\omega^2) \quad \text{in QG.} \quad (51)$$

We drop in the sequel the tilde and write $A_\mu, A_\mu^a, g_{\mu\nu}$ instead of $\tilde{A}_\mu, \tilde{A}_\mu^a, \tilde{g}_{\mu\nu}$. We treat separately the cases of QED, YM and QG.

– QED. From $\delta_\xi A_\mu(p) = ip_\mu \xi(p)$ (eq. (26)), and eq. (43), from differentiating w.r.t. q it immediately follows

$$p_{\mu_0}^0 \Gamma_{n+1}^{\mu_0 e^1 \dots e^n}(p^0, p^1, \dots, p^n) = 0 \quad (52)$$

(we have factored out $\delta^4(\sum_{i=0}^n p^i)$), whence formula (49) follows at once (using boson symmetry), if we choose p^i so that the sets $\{p^0, \dots, p^n\}, \{k^0, \dots, k^n\}$ coincide. Actually we can derive directly from eq. (39) the stronger property

$$k_{\mu_i}^i \Gamma_{n+1}^{\mu_0 \dots \mu_i \dots \mu_n}(k^0, \dots, k^i, \dots, k^n) = 0, \quad n \geq 2 \quad (53)$$

– YM. From

$$\delta_\xi A_\mu^a(p) = ip_\mu \xi^a(p) + e f^{abc} \int d^4 q A_\mu^b(p-q) \xi^c(q) \quad (54)$$

(eq. (27) in momentum space), and from differentiating formula (43) (with $n = m$) w.r.t. $\xi(p^0)$, it immediately follows

$$ip_{\mu_0}^0 \Gamma_{m+1}^{\mu_0 a_0, e^1 a_1, \dots, e^m a_m}(p^0, p^1, \dots, p^m) + \sum_{l=1}^m e f^{b_l a_l a_0} \Gamma_m^{e^1 a_1, \dots, e^{l-1} a_{l-1}, e^l b_l, e^{l+1} a_{l+1}, \dots, e^m a_m}(p^1, \dots, p^{l-1}, p^l + p^0, p^{l+1}, \dots, p^m) = 0 \quad (55)$$

(again, we have factored out $\delta^4(\sum_{i=0}^m p^i)$). This formula holds for any configuration $\sum_{i=0}^m p^i = 0$, $e^i(p^i) \cdot p^i = 0$. On a configuration ω -converging to the decay configuration we deduce from the induction hypothesis that the second term is $o(\omega^{4-m})$.

– QG. The gauge transformation (28) in momentum space reads

$$\delta_\xi g_{\mu\nu}(p) = i \int d^4r \left\{ g_{\rho\nu}(p-r) r_\mu \xi^\rho(r) + g_{\rho\mu}(p-r) r_\nu \xi^\rho(r) + \xi^\rho(p-r) r_\rho g_{\mu\nu}(r) \right\}, \quad (56)$$

implying

$$\delta_\xi g_{\mu\nu}(p)|_{g_{\mu\nu}(p)=\eta_{\mu\nu}\delta^4(p)} = i \{ p_\mu \xi^\rho(p) \eta_{\rho\nu} + p_\nu \xi^\rho(p) \eta_{\rho\mu} \}. \quad (57)$$

Moreover, we note that

$$\frac{\delta g_{\alpha\beta}(p)}{\delta g_{\mu\nu}(-q)} = \delta^4(p+q) \left[\delta_\alpha^\mu \delta_\beta^\nu + \delta_\alpha^\nu \delta_\beta^\mu \right]. \quad (58)$$

After differentiation w.r.t. $\xi^{\nu_0}(p^0)$, Eq. (43) with $n = m$ reads:

$$\begin{aligned} 0 &= \Gamma_{m+1}^{\mu\nu, e^1, \dots, e^m}(p^0, p^1, \dots, p^m) 2(p^0)_\mu \eta_{\nu\nu_0} \\ &+ \sum_{h=1}^m \left[\Gamma_m^{e^1 \dots e^{h-1}, \mu\nu, e^{h+1} \dots e^n}(\dots, p^{h-1}, p^0 + p^h, p^{h+1}, \dots) 4(p^0)_\mu (\varepsilon^h)_\nu (\varepsilon^h)_{\nu_0} \right. \\ &\quad \left. + (p^h)_{\nu_0} \Gamma_m^{e^1 \dots e^n}(\dots, p^{h-1}, p^0 + p^h, p^{h+1}, \dots) \right] \Big|_{g_{\alpha\beta}(p)=\eta_{\alpha\beta}\delta^4(p)}. \end{aligned} \quad (59)$$

(once again, we have factored out $\delta^4(\sum_{i=0}^m p^i)$). This formula holds for any configuration $\sum_{i=0}^m p^i = 0$, $e^i(p^i) \cdot p^i = 0$. On a configuration ω -converging to the decay configuration we deduce from the induction hypotheses (51), (48) that the second, third terms are $o(\omega^2)$, which proves eq. (51) for $n = m$.

Part 2: We prove the factorization formulae

$$\Gamma_{n+1}^{e^0 \dots e^n}(k^0, \dots, k^n) = \sum_P A_{i_0 i_1 \dots i_n} E^{i_0 i_1} \dots E^{i_{n-1} i_n} \quad \text{in QED, } (n+1) \text{ even;} \quad (60)$$

$$\Gamma_{n+1}^{e^0 a_0 \dots e^n a_n}(k^0, \dots, k^n) = \sum_P A_{i_0 i_1 \dots i_n}^{a_0 \dots a_n} E^{i_0 i_1} \dots E^{i_{n-1} i_n} + o(\omega^{3-n}) \quad \text{in YM, } (n+1) \text{ even;} \quad (61)$$

$$\Gamma_{n+1}^{e^0 \dots e^n}(k^0, \dots, k^n) = \sum_P A_{i_0 i_1 \dots i_{2n} i_{2n+1}} E^{i_0 i_1} \dots E^{i_{2n} i_{2n+1}} + o(\omega) \quad \text{in QG.} \quad (62)$$

and

$$\Gamma_{n+1}^{e^0 \dots e^n}(k^0, \dots, k^n) = \sum_P \sum_{j_0=0}^n A_{i_0 i_1 \dots i_n}^{j_0} (k^{j_0} \cdot \varepsilon^{i_0}) E^{i_1 i_2} \dots E^{i_{n-1} i_n} \quad \text{in QED, } (n+1) \text{ odd;} \quad (63)$$

$$\Gamma_{n+1}^{e^0 a_0 \dots e^n a_n}(k^0, \dots, k^n) = \sum_P \sum_{j_0=0}^n A_{i_0 i_1 \dots i_n}^{j_0; a_0 \dots a_n} (k^{j_0} \cdot \varepsilon^{i_0}) E^{i_1 i_2} \dots E^{i_{n-1} i_n} + o(\omega^{3-n}) \quad \text{in YM, } (n+1) \text{ odd;} \quad (64)$$

where:

1) \sum_P means the sum over all the permutations P ($P \equiv (i_0, i_1, \dots, i_n)$ is a permutation of $(0, 1, \dots, n)$ in QED and YM, whereas $P \equiv (i_0, i_1, \dots, i_{2n+1})$ is a permutation of $(0, 1, \dots, 2n+1)$ in QG);

2) the A 's are scalar functions depending on the scalar products $k^i \cdot k^j$ (and, in the Y.M. case, on $2m$ Lie algebra indices a_i);

3) we have introduced the shorthand notation

$$E^{ij} := \left(\varepsilon^i \cdot \varepsilon^j k^i \cdot k^j - \varepsilon^i \cdot k^j \varepsilon^j \cdot k^i \right). \quad (65)$$

In the RHS of eq. (62) it is tacitly understood that $\varepsilon^{2s+1} \equiv \varepsilon^{2s}$, $k^{2s+1} = k^{2s}$, $s = 0, \dots, 2n$.

We prove explicitly the first three (the proof of formulae (63), (64), is completely analogous): let $n+1 = 2m$. We look for the most general $\Gamma_{n+1}^{\mu_1 \dots \mu_{n+1}}(k^1, \dots, k^{n+1})$ satisfying:

1) the constraint

$$\Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^{i-1}, \mu_i, \varepsilon^{i+1} \dots \varepsilon^n}(k^0, \dots, k^n) k_{\mu_i}^i = o(\omega^d) \quad (66)$$

in any configuration $(k^i(\omega), \varepsilon^i(\omega))_{i=0, \dots, n}$ ω -converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0, \dots, n}$;

2) symmetry under any replacement $(\mu_i, k^i) \leftrightarrow (\mu_l, k^l)$, $i, l = 0, \dots, n$.

If we set $o(\omega^d) \equiv 0$ this amounts to solving eq. (49) equipped with boson symmetry for the $(n+1)$ -photons vertex function of Q.E.D.; if we set $d = 3 - n$, this amounts to solving eq. (50) equipped with boson symmetry for the $(n+1)$ -gluons vertex function of Y.M., provided we understand an implicit dependence of Γ_{n+1} on the Lie algebra indices a_i and remind that the latter have to be permuted along with the indices μ_i and the momenta k^i when boson symmetry is imposed; if we choose $n+1 = 4r$, $d = 2$, and add the additional symmetry conditions $k^{2i+1} = k^{2i}$, $\varepsilon^{2i+1} = \varepsilon^{2i}$ ($i = 0, \dots, 2r - 1$), this will amount to solving eq. (51) equipped with boson symmetry for the $2r$ -gravitons vertex function of Q.G. In this way, we can formally deal with eq.'s (49), (50), (51) simultaneously, by just dealing with one.

The dependence of $\Gamma_{n+1}^{\mu_0 \dots \mu_n}(k^0, \dots, k^n)$ on Lorentz indices can only occur through the metric tensors $\eta^{\mu_i \nu_j}$ and the 4-vectors k^{μ_i} . Compactly, the most general dependence can be written in the following way

$$\Gamma_{n+1}^{\mu_0 \dots \mu_n} = \sum B^0 \underbrace{\eta \dots \eta}_{m \text{ times}} + \sum B^1 k k \underbrace{\eta \dots \eta}_{(m-1) \text{ times}} + \dots + \sum B^m \underbrace{k \dots k}_{2m \text{ times}}, \quad (67)$$

where the B 's denote Lorentz scalar functions. For our purposes, it will be more convenient to expand Γ_{n+1} in terms of the 4-vectors k^{μ_i} and of the tensors $E^{\mu_i \mu_j}(k^i, k^j) := \eta^{\mu_i \mu_j} k^i \cdot k^j -$

$(k^i)^{\mu_j} (k^j)^{\mu_i}$), which satisfy the relation

$$(k^i)_{\mu_i} E^{\mu_i \mu_j} = 0 = (k^j)_{\mu_j} E^{\mu_i \mu_j} \quad (68)$$

The general expansion (67) can be replaced by

$$\Gamma_{n+1}^{\mu_0 \dots \mu_n} (k^0, \dots, k^n) = \sum_P \sum_{l=0}^m \sum_{j_0, \dots, j_{2l-1}=0}^n A_{i_0 \dots i_n}^{l; j_0 \dots j_{2l-1}} (k^{j_0})^{\mu_{i_0}} \dots (k^{j_{2l-1}})^{\mu_{i_{2l-1}}} E^{\mu_{i_{2l}} \mu_{i_{2l+1}}} \dots E^{\mu_{i_{n-1}} \mu_{i_n}} \quad (69)$$

where \sum_P means the sum over all the permutations $P \equiv (i_0, i_1, \dots, i_n)$ of $(0, 1, \dots, n)$ and $A_{i_0 \dots i_n}^{l; j_1 \dots j_{2l}}$ are scalar functions depending on the scalar products $k^i \cdot k^j$ (and, in the Y.M. case, on $2m$ Lie algebra indices a_i).

We have introduced a quite redundant set of scalars $\{A_{i_0 \dots i_n}^{l; j_1 \dots j_{2l}}\}$ to make formula (69) more compact. The set is redundant in the sense that $A_{i_0 \dots i_n}^{l; j_0 \dots j_{2l-1}}$ and $A_{i_0 \dots i_n}^{l; \hat{j}_0 \dots \hat{j}_{2l-1}}$ will both contribute to the same term $(k^{j_0})^{\mu_{i_0}} \dots (k^{j_{2l-1}})^{\mu_{i_{2l-1}}} E^{\mu_{i_{2l}} \mu_{i_{2l+1}}} \dots E^{\mu_{i_{n-1}} \mu_{i_n}}$ in the expansion (69), whenever

1) there exists a permutation P_{2l} of $2l$ objects such that $(\hat{i}_0, \hat{i}_1, \dots, \hat{i}_{2l-1}) = P_{2l}(i_0, i_1, \dots, i_{2l-1})$, $(\hat{j}_0, \hat{j}_1, \dots, \hat{j}_{2l-1}) = P_{2l}(j_0, j_1, \dots, j_{2l-1})$;

2) $(\hat{i}_{2l}, \hat{i}_{2l+1}, \dots, \hat{i}_n) = P_{n+1-2l}(i_{2l}, i_{2l+1}, \dots, i_n)$, where P_{2m-2l} is a permutation of $n+1-2l = 2m-2l$ objects which is the product: 2.a) of transpositions between the $(2s)^{th}$ and the $(2s+1)^{th}$ object ($s = 1, \dots, m-l$); 2.b) of transpositions between different pairs $(2s, 2s+1)$, $(2r, 2r+1)$, $r, s = 1, \dots, m-l$.

We are free to set $A_{i_0 \dots i_n}^{l; j_0 \dots j_{2l-1}} = A_{i_0 \dots i_n}^{l; \hat{j}_0 \dots \hat{j}_{2l-1}}$ in these cases.

Finally, boson symmetry (36) implies that the scalars A^l satisfy the relations

$$A_{\dots j \dots i \dots}^{l; \tilde{j}_0 \dots \tilde{j}_{2l-1}} (k^i \leftrightarrow k^j) = A_{\dots i \dots j \dots}^{l; j_0 \dots j_{2l-1}} \quad \tilde{h} := \begin{cases} j & \text{if } h = i \\ i & \text{if } h = j \\ h & \text{if } h \neq i, j \end{cases} \quad (70)$$

for any pair of indices i, j .

Plugging the general expansion (69) into Eq. (66) and using relation (68) we find

$$o(\omega^d) = \sum_{P'} \sum_{l=1}^m \sum_{j_0, \dots, j_{2l-1}=0}^n \left[A_{i_1 \dots i_n}^{l; j_1 \dots j_{2l}} + A_{i_1 \dots i_n}^{l; j_1 \dots j_{2l}} + \dots + A_{i_1 \dots i_{2l-1} i_{2l} \dots i_n}^{l; j_0 \dots j_{2l-1}} \right] \times (k^i \cdot k^{j_0}) \varepsilon^{i_1} \cdot k^{j_1} \dots \varepsilon^{i_{2l-1}} \cdot k^{j_{2l-1}} E^{i_{2l} i_{2l+1}} \dots E^{i_{n-1} i_n}, \quad (71)$$

where $\sum_{P'}$ means the sum over all the permutations $P' \equiv (i_1, \dots, i_n)$ of $(0, 1, \dots, i-1, i+1, \dots, n)$, whereas

$$\Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^n} (k^0, \dots, k^n) = \sum_{P'} \sum_{l=0}^m \sum_{j_0, \dots, j_{2l-1}=0}^n \left[A_{i_1 \dots i_n}^{l; j_0 \dots j_{2l-1}} + A_{i_1 \dots i_n}^{l; j_0 \dots j_{2l-1}} + \dots + A_{i_1 \dots i_{2l-1} i_{2l} \dots i_n}^{l; j_0 \dots j_{2l-1}} \right] \times (\varepsilon^i \cdot k^{j_0}) \varepsilon^{i_1} \cdot k^{j_1} \dots \varepsilon^{i_{2l-1}} \cdot k^{j_{2l-1}} E^{i_{2l} i_{2l+1}} \dots E^{i_{n-1} i_n}. \quad (72)$$

Note that the term $l = 0$ has completely disappeared from the sum in eq. (71), due to eq. (68).

Let us fix the xyz axes so that $k^0 = (k_0^0, 0, 0, k_0^0)$ [according to property 2 this implies $k^j = \lambda^j(k_0^0, 0, 0, k_0^0)$, $j = 1, 2, \dots, n$]; we can always assume that the polarization vectors $\hat{\varepsilon}^i$ are real and have the form $\hat{\varepsilon}^i = (0, \cos\theta^i, \sin\theta^i, 0)$. We now start exploiting the available freedom in the choice (1) of the angles θ^i characterizing the polarization vectors $\hat{\varepsilon}^i$; (2) of the configuration (k^i, ε^i) ω -converging to $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0, \dots, n}$. A family of possible choices of the latter is

$$\begin{aligned} k^i &\equiv \hat{k}^i + \omega b^i \hat{\varepsilon}^i & \hat{\varepsilon}^i &:= (0, -\sin\theta^i, \cos\theta^i, 0) & i &= 0, 1, \dots, n, \\ \varepsilon^i &\equiv \hat{\varepsilon}^i; \end{aligned} \quad (73)$$

the family is parametrized by the $2n + 2$ parameters (b^i, θ^i) , which are only constrained by the condition $\sum_{i=0}^n b^i \hat{\varepsilon}^i = 0$ (so that $\sum_{i=0}^n k^i = \sum_{i=0}^n \hat{k}^i = 0$). As a consequence

$$k^i \cdot k^j = -\omega^2 b^i b^j \cos(\theta^i - \theta^j), \quad \varepsilon^i \cdot k^j = -\omega b^j \sin(\theta^i - \theta^j) \quad \varepsilon^i \cdot \varepsilon^j = -\cos(\theta^i - \theta^j) \quad (74)$$

$$\left(\varepsilon^i \cdot \varepsilon^j k^i \cdot k^j - \varepsilon^i \cdot k^j \varepsilon^j \cdot k^i \right) = \omega^2 b^i b^j \quad (75)$$

By plugging these (k^i, ε^i) into Eq. (71) we find

$$\begin{aligned} o(\omega^d) &= \omega^{n+2} \sum_{P'} \sum_{l=1}^m \sum_{j_0, \dots, j_{2l-1}=0}^n \sum_{i=1}^n \left[A_{i_1 \dots i_n}^{l; j_0 \dots j_{2l-1}} + A_{i_1 i \dots i_n}^{l; j_0 \dots j_{2l-1}} + \dots + A_{i_1 \dots i_{2l-1} i_{2l} \dots i_n}^{l; j_0 \dots j_{2l-1}} \right] \\ &b^i b^{j_0} \dots b^{j_{2l-1}} b^{i_{2l}} \dots b^{i_n} \cos(\theta^i - \theta^{j_0}) \sin(\theta^{i_1} - \theta^{j_1}) \dots \sin(\theta^{i_{2l-1}} - \theta^{j_{2l-1}}). \end{aligned} \quad (76)$$

The coefficients in the square brackets can depend on the angles θ^i only through the cosines $\cos(\theta^i - \theta^j)$ (since $k^i \cdot k^j = -\omega^2 b^i b^j \cos(\theta^i - \theta^j)$); since the above equation has to hold for all θ^i 's then all terms in the square brackets have to satisfy the relation

$$\left[A_{i_1 \dots i_n}^{l; j_0 \dots j_{2l-1}} + A_{i_1 i \dots i_n}^{l; j_0 \dots j_{2l-1}} + \dots + A_{i_1 \dots i_{2l-1} i_{2l} \dots i_n}^{l; j_0 \dots j_{2l-1}} \right] = o(\omega^{d-n-2}) \quad l = 1, \dots, m \quad (77)$$

independently.

Replacing the above results in formula (72) we find the factorization formula

$$\Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^n} (k^0, \dots, k^n) = \sum_P A_{i_0 i_1 \dots i_n}^{0; } E^{i_0 i_1} \dots E^{i_{n-1} i_n} + o(\omega^{d-1}) \quad (78)$$

whence formulae (60), (61), (62) follow.

Part 3: On any configuration $(k^i(\omega), \varepsilon^i(\omega))_{i=0, \dots, n}$ ω -converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0, \dots, n}$ we have $E^{ij} = o(\omega^2)$. To prove formulae (45), (46), (48) it remains to show that the

scalar functions A 's appearing in eq.'s (60), (61), (62) can show poles in ω at most of degree so high to yield the global ω -dependence reported in the former formulae. For this purpose we use a continuity argument, i.e. we argue that the claimed ω -dependence is the only one compatible with equations (60), (61), (62) if we require the LHS to be independent of the particular configuration $(k^i(\omega), \varepsilon^i(\omega))_{i=0,\dots,n}$ ω -converging to the decay configuration $(\hat{k}^i, \hat{\varepsilon}^i)_{i=0,\dots,n}$.

For the sake of brevity we continue to use the factorization formula (78) to deal at once with all three cases. We choose two different multi-parameter families $(k^i(\omega), \varepsilon^i(\omega))$, $(\tilde{k}^i(\omega), \tilde{\varepsilon}^i(\omega))$ of configurations ω -converging to the decay configuration, and we require that

$$\lim_{\omega \rightarrow 0} \Gamma_{n+1}^{\varepsilon^0(k^0)\dots\varepsilon^n(k^n)}(0, \dots, k^n) = \lim_{\omega \rightarrow 0} \Gamma_{n+1}^{\tilde{\varepsilon}^0(k^0)\dots\tilde{\varepsilon}^n(k^n)}(\tilde{k}^0, \dots, \tilde{k}^n) \quad (79)$$

In the xyz axes as before, the first is the family (73), the second is

$$\tilde{k}^i := \hat{k}^i + \omega(0, 0, c^i, 0) \quad \tilde{\varepsilon}^i := [(\hat{k}_3^i)^2 + (\omega c^i)^2]^{-\frac{1}{2}}(0, 0, \hat{k}_3^i, \omega c^i) \quad i = 0, 1, \dots, n, \quad (80)$$

where $\sum_{i=0}^n c^i = 0$. This implies in particular $\tilde{k}^i \cdot \tilde{k}^j = -\omega^2 c^i c^j$.

With the first family we find

$$\Gamma_{n+1}^{\varepsilon^0\dots\varepsilon^n}(k^0, \dots, k^n) = \omega^{n+1} b^0 \dots b^n \sum_P A_{i_0 i_1 \dots i_n}^{0; \dots} + o(\omega^{d-1}). \quad (81)$$

Now we specialize our discussion to the case of QED and QG, where $d - 1 \geq 1$, so that the second term vanishes when $\omega \rightarrow 0$. Let us consider *per absurdum* the hypothesis that the functions A 's have poles of degree $(n+1)$ in ω . In order that the RHS has a limit independent of the b^i 's when $\omega \rightarrow 0$, the A 's must have the form

$$A_{i_0 i_1 \dots i_n}^{0; \dots} = \left[\sum_P a_{i_0 i_1 \dots i_n} k^{i_0} \cdot k^{i_1} \dots k^{i_{n-1}} \cdot k^{i_n} \right]^{-1}, \quad (82)$$

where $a_{i_0 i_1 \dots i_n}$ are constants, so that

$$A_{i_0 i_1 \dots i_n}^{0; \dots} = \left[\omega^{n+1} b_0 \dots b^n \right]^{-1} \times const. \quad (83)$$

On the other hand, plugging the family (80) into eq. (82) and replacing the result into formula (78), we find

$$\Gamma_{n+1}^{\tilde{\varepsilon}^0(k^0)\dots\tilde{\varepsilon}^n(k^n)}(\tilde{k}^0, \dots, \tilde{k}^n) = const. \times \sum_P \left(\frac{d^{i_0}}{d^{i_1}} + \frac{d^{i_1}}{d^{i_0}} - 1 \right) \dots \left(\frac{d^{i_{n-1}}}{d^{i_n}} + \frac{d^{i_n}}{d^{i_{n-1}}} - 1 \right) + o(\omega^2) \quad (84)$$

where we have defined $d^i := \frac{k_3^i}{c^i}$. This expression depends on the choice of the coefficients c^i , i.e. depends on the way the family $(\tilde{k}^i(\omega), \tilde{\varepsilon}^i(\omega))$ approaches $(\hat{k}^i(\omega), \hat{\varepsilon}^i(\omega))$, against the hypothesis.

In a similar way, one can exclude the hypothesis that the functions A 's have poles in ω of degree $> (n+1)$, otherwise the RHS would diverge to either $+\infty$ or $-\infty$ according to the way the families approach the decay configuration

Summing up, we have discarded the possibility that the A 's have poles in ω of degree $\geq n+1$, so that consequently in QED, QG

$$\Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^n}(k^0, \dots, k^n) = o(\omega) \quad (85)$$

In QED we can improve the bound (85) into the stronger bound (45). In fact, if one plugs the general expansion (69) into eq. (53) [instead of eq. (66)] and argues as in part 2, one ends up with a stronger form of the factorization,

$$\Gamma_{n+1}^{\mu_0 \dots \mu_n}(k^0, \dots, k^n) = \sum_P A_{i_0 i_1 \dots i_n} E^{\mu_{i_0} \mu_{i_1}} \dots E^{\mu_{i_{n-1}} \mu_{i_n}}. \quad (86)$$

Looking at the Feynman diagrams contributing to each order in the loops to $\Gamma_{n+1}^{\mu_0 \dots \mu_n}(k^0, \dots, k^n)$, it is easily realized that they are continuous and finite for all values of k^i 's, since the fermion/scalar masses are infrared cutoffs [see fig. (3)]. Hence, the scalars A cannot have poles in $k^i \cdot k^j$, because otherwise at least the terms $A_{i_0 i_1 \dots i_n} (k^{i_0})^{\mu_{i_1}} \dots (k^{i_n})^{\mu_{i_{n-1}}} (k^{i_{n-1}})^{\mu_{i_n}}$ would diverge. The A 's have dimension $[mass]^{4-2(n+1)}$, since Γ_{n+1} has dimension $[mass]^{4-(n+1)}$. This can be accounted for without introducing poles in $k^i \cdot k^j$, but using the mass parameters of the charged particle interacting with the photon. For instance, if the only charged particle is a fermion with mass m , then $A = m^{4-2(n+1)} o(1)$. We have completed the proof of the claim (45).

In QG the $o(\omega)$ in the RHS of (85) can be improved into a $o(\omega^2)$, since $\Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^n}(k^0, \dots, k^n)$ can be only of even degree in ω , if we assume that the proper vertices depend analitically on the momenta k^i . This follows from formula (44), because the LHS of eq. (85) has to be a function of the Lorentz scalars $k^i \cdot k^j$, $\varepsilon^i(k^i) \cdot k^j$, of even degree in the latter. This completes the proof of the claim (48).

In YM formula (78) and the continuity argument do not exclude that there exists a limit $\lim_{\omega \rightarrow 0} \Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^n}(k^0, \dots, k^n) =: L \neq 0$ independent of the way the family $(k^i(\omega), \varepsilon^i(\omega))$ approaches $(\hat{k}^i(\omega), \hat{\varepsilon}^i(\omega))$. In fact, if the functions A 's have a pole of degree $\geq (n+1)$ in ω , the second term in formula (78) (which in principle can be finite or divergent) could compete with the first, and $\Gamma_{n+1}^{\varepsilon^0 \dots \varepsilon^n}(k^0, \dots, k^n)$ could have a family-independent limit even though the first term has not. This is exactly what happens with the 4-gluon proper vertex, as one can already check at the tree level

$$\Gamma_{4,tree}^{\hat{\varepsilon}^0 a_0 \dots \hat{\varepsilon}^3 a_3} \propto [(\hat{\varepsilon}^0 \cdot \hat{\varepsilon}^1)(\hat{\varepsilon}^2 \cdot \hat{\varepsilon}^3) f^{a_0 a_3 e} f^{a_1 a_2 e} + perm.] \neq 0. \quad (87)$$

By an explicit analysis of the general expansion (69) one can easily realize that a family-independent limit $L \in \mathbf{R} \cup \{\pm\infty\}$ can be obtained only if equation (46) is satisfied.

Finally, the proof of the general claim (47) can be done by an induction procedure in the number of external photons (resp. of YM bosons) which mimics the one sketched so far for QED (resp. YM), with the only difference that as starting input we do not use the value of proper vertex with zero photons, zero YM bosons and zero gravitons, but the proper vertex with $m_g > 0$ gravitons or $m_y > 0$ YM bosons (resp. with $m_g > 0$ gravitons or $m_\gamma > 0$ photons).

We have thus completed the proof of property 10 \diamond .

5 Concluding remarks.

We have seen that the decay probabilities for the photon, the graviton and the Yang-Mills boson all vanish (perturbatively). The decay amplitudes involving only photons and/or gravitons are themselves zero; we have first shown these properties by a simple power counting argument and then proved them rigorously through the Ward identities, assuming only continuity of the Greens functions in the infrared limit. In the case of the Yang-Mills boson, the amplitude does not vanish in the infrared limit (more precisely, it diverges if $m \geq 5$ out of $n+1$ external particles are YM bosons); the decay probability is however suppressed by the phase-space factor. The latter is the only case in which we have needed an infrared regulator.

In this final Section we would like to comment on the relation between our work and the classical literature [2] on infrared divergences in quantum field theory.

For the reasons just mentioned, even in YM theories we do not need to average (*à la Bloch and Nordsieck* [2]) over sets of states degenerate in the energy, like in the Kinoshita-Lee-Nauenberg theorem [2], in order to build finite physical transition probabilities out of divergent amplitudes.

However, this might be necessary for other theories, not explicitly considered here, where the divergences of the amplitudes are sufficiently bad. In the latter case it would be nevertheless important to keep in mind some peculiarities of the decay of massless particles compared to what one usually finds in the literature [2]. The physical processes explicitly considered in the literature are either scatterings, or decays in which the initial particle is massive. The case of a decay process where all particles (including the initial one) are massless is not considered. In a theory including massless particles, the Green functions may diverge if (1) some of the external particles are soft/collinear massless ones and/or (2) if massless particles appear among

the internal ones occurring in the corresponding Feynman diagrams (e.g. in loops). The study of these divergences is usually performed by attributing a small mass m to each kind of massless particle in the theory and then studying the limit in which m goes to zero (as already recalled, their elimination from "physical" transition probabilities is obtained by building up the initial and/or final states as a mixture of degenerate states of the energy before performing the limit; see Ref.'s [2]).

On the other hand we note that, while in the scattering processes or in the decay processes of a massive particle the collinearity of some massless external particles is one of the kinematically allowed configuration [so that it makes sense to study the divergences of the Green functions in the limit when these external momenta become collinear while remaining on-shell (null)], in the decay of a massless particle the only allowed kinematical configuration is that in which all the external particles are collinear. Therefore, the divergences, if they occur, characterize all the kinematically allowed configurations. Moreover, the latter typically appear already at the tree-level (consider e.g. T_5 for the YM theory), as one can immediately check by summing all relevant tree diagrams. As for the IR regulator, the one based on the attribution of a small mass to the external particles is unsuitable because it forbids the decay of a particle into other ones of the same kind (as we have already noted in section 1.3). Our regulator, based on a small-frequency external source, bypasses this difficulty while having the nice feature of being physically intuitive.

As mentioned at the end of Section 2, a partial decay probability Γ_n different from zero can be only obtained when the square amplitude is proportional to a sufficiently high negative power of ω . If we admit (as is generally true in perturbation theory) that the coupling constants appear in the numerator, this means that the amplitude must contain a coupling constant with positive mass dimension.

One of the few theories we are aware of, in which such a coupling occurs (besides the $\lambda\phi^3$ theory; compare Section 1) is gravity in the presence of a cosmological constant. In this case the action of the gravitational field is written as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g(x)} [\Lambda - R(x)] \quad (88)$$

or, redefining the metric in the form $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa\tilde{h}_{\mu\nu}(x)$, with $\kappa = \sqrt{16\pi G}$,

$$S = \int d^4x \sqrt{1 + \kappa\tilde{h} + \kappa^2\tilde{h}^2 + \kappa^3\tilde{h}^3 + \dots} \left[\frac{\Lambda}{\kappa^2} - \tilde{R}^{(2)}(x) + \dots \right] \quad (89)$$

We have denoted symbolically with \tilde{h} , \tilde{h}^2 , \tilde{h}^3 ... in the square root terms which are linear, quadratic, cubic ... in \tilde{h} , omitting the indices and the exact algebraic structure. $\tilde{R}^{(2)}(x)$ denotes

the part of the curvature quadratic in \tilde{h} . The term $\kappa^3\tilde{h}^3$, when is multiplied by Λ/κ^2 , gives rise to a vertex $\kappa\Lambda\tilde{h}^3$ which couples three gravitons with a coupling constant $\kappa\Lambda$ of mass dimension 1 (unlike the corresponding three-vertex of the pure Einstein action, which is proportional to κ^3 and contains 4 four-momenta, so that the infrared processes are strongly suppressed).

Although the decay amplitudes involving this new three-vertex are suppressed at the tree-level because of helicity conservation (Property 3), it can be used to construct gravitonic loops with n external legs. The amplitudes will be proportional to positive powers of $\kappa\Lambda$ and – in our regularization scheme – to negative powers of ω . This means that Γ_n would be finite in the limit $\omega \rightarrow 0$, or even diverge. But we should not forget the terms which are linear and quadratic in \tilde{h} in the square root of eq. (89). In particular, the quadratic term gives rise to a graviton mass (if $\Lambda < 0$) or to instability (if $\Lambda > 0$) [6]. In the first case, we end up with gravitons which are not massless any more, so that all our preceding formalism does not apply.

It is known that the cosmological constant Λ , although possibly very large in principle, is limited by astronomical observations to be less than $|\Lambda| \leq 10^{120}G^{-1}$ (in order to explain this vanishing, many mechanisms have been proposed [14]). Therefore it seems that the idea of a decay induced by the presence of a cosmological constant can be excluded on the basis of the empirical evidence.

However, in the non-perturbative quantum Regge calculus [15] the effective value of the adimensional product $|\Lambda|G$ depends on the length scale and vanishes according to a power law as the energy scale μ goes to zero:

$$|\Lambda|G \sim (l_0\mu)^\gamma \tag{90}$$

If we admit that the average lattice spacing l_0 is of the order of the Planck length [3], then the constant Λ can be non-vanishing on small scales, leaving the graviton massless at large scales. This might change the situation, but clearly at the present stage of knowledge these are still speculative hypotheses.

We wish to thank M. Abud, P. Van Nieuwenhuizen and G. A. Vilkovisky for useful discussions, as well as the A. Von Humboldt foundation for financial support. We are grateful to D. Maison and J. Wess for the kind hospitality at their institutes.

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Appendix

We prove eq. (41). In the case of QED with (for instance) Feynman's gauge-fixing $\frac{1}{2\alpha} \int d^4x (\partial_\mu A_\mu)^2$, the LHS is zero when $n > 1$ because the gauge variation of the gauge-fixing above is of first degree in A_μ , and is zero in the case $n = 1$ because

$$-\frac{1}{\alpha} p_1^\mu (p_1)^\nu \xi(p_1), \quad (91)$$

vanishes after contraction with the polarization vector $\varepsilon_\mu^\pm(p_1)$. In the case of YM with (for instance) Feynman's gauge-fixing $\frac{1}{\alpha} \int d^4x (\partial^\mu A_\mu)^2$, the LHS is zero if $n > 2$ because the gauge variation of the gauge-fixing above is of second degree in A_μ ; if $n = 1$ it is zero for the same reason as in the preceding case (91); if $n = 2$ it is zero because

$$-\frac{2}{\alpha} (p_1)^{\mu_1} (p_2)^{\mu_2} \xi^c(p_1 + p_2) f^{a_1 a_2 c}, \quad (92)$$

vanishes after contractions with the polarization vector $\varepsilon_{\mu_1}^\pm(p_1) \varepsilon_{\mu_2}^\pm(p_2)$. In the case of QG with harmonic gauge-fixing $\frac{1}{2\alpha} \int d^4x (\partial^\mu h_{\mu\nu})^2$ we have

$$\delta_\xi \left[\frac{1}{2\alpha} \int d^4x (\partial^\mu h_{\mu\nu})^2 \right] = -\frac{1}{\alpha} \int d^4p (p^\mu h_{\mu\nu}(p)) p^\rho (\hat{\xi}_{;\rho}^\nu + \hat{\xi}_\rho^{\nu'})(-p) \quad (93)$$

(here $\hat{}$ means Fourier transform). When some $\frac{\delta}{\delta g_{\mu_i \nu_i}(-p_i)}$ acts on $p^\mu h_{\mu\nu}(p)$ we get a factor $\delta^4(p + p_i) [\delta_\mu^{\mu_i} \delta_\nu^{\nu_i} + \delta_\nu^{\mu_i} \delta_\mu^{\nu_i}]$ (see formula (58)), which gives zero after contraction with the polarization tensor $e^{\mu_i \nu_i}(p_i)$. \diamond