

Incompatibility Semantics from Agreement

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Abstract In this paper, I discuss the analysis of logic in the pragmatic approach recently proposed by Brandom. I consider different consequence relations, formalized by classical, intuitionistic and linear logic, and I will argue that the formal theory developed by Brandom, even if provides powerful foundational insights on the relationship between logic and discursive practices, cannot account for important reasoning patterns represented by non-monotonic or resource-sensitive inferences. Then, I will present an incompatibility semantics in the framework of linear logic which allow to refine Brandom's concept of defeasible inference and to account for those non-monotonic and relevant inferences that are expressible in linear logic. Moreover, I will suggest an interpretation of discursive practices based on an abstract notion of agreement on what counts as a reason which is deeply connected with linear logic semantics.

1 Introduction

The analytical pragmatism proposed by Brandom, stressing that “we must look at what it is to use locutions as expressing meanings – that is, at what one must do in order to count as saying what the vocabulary lets the practitioners express” [5]), sheds new lights on foundational issues concerning logic, in particular, developing a strong connection between logic and inferential (pre-logical) practice or abilities, which are to be grounded in the general practice of giving and asking for reasons.

I will investigate different logical vocabularies as related to different aspects of a pre-logical inferential practice and I will challenge Brandom's model, which is based on a particular incompatibility semantics, taking into account two well-established non-classical paradigms of reasoning, represented by intuitionistic and linear logic, for which an account in Brandom's terms has not yet been provided.

I will argue that Brandom's formal model cannot find a suitable place for intuitionistic, causal or non-monotonic reasoning and I will claim that this is a severe drawback for the generality of Brandom's theory. Then, I will propose a different model of incompatibility semantics, based on the algebraic structure providing models of linear logic, that can express classical, intuitionistic and linear consequence relations in a harmonious way. In particular, I will investigate a refinement of Brandom's notion of

defeasibility, a condition required in order to define consequence relations from incompatibility relations, which may be used to keep track of some instances of relevant reasoning. From the technical point of view, the formal approach here presented can be considered a generalization of Brandom’s semantics towards non-monotonic, intuitionistic and resource-sensitive reasoning, since all the definition I will present can be rephrased to fit in the formal apparatus defined in [5]. However, I will provide a different justification for incompatibility semantics based on an abstract form of agreement, a deal between a proponent and an opponent in a dialog, on what count as a reason. The intuition leading the approach I am proposing is that a pragmatic account of the meaning of logical constants, suggesting that we should look at what it is *to use* logical connectives as expressing logical meanings, requires an interactive point of view on logic, since, as Wittgenstein puts it “it is not possible to obey a rule ‘privately’ ”¹. The notion of incompatibility, which is defined by Brandom as a constitutively modal notion, will be here interpreted using the concept of agreement on “what count as incompatible” in a given context.

In Section 2, I will recall Brandom’s main definitions of meaning-use relations. In Section 3, I will discuss Brandom’s incompatibility semantics and I will investigate its relationship with intuitionistic and linear inferences. Section 4 contains a succinct presentation of linear logic and its expressive power in order to show its advantages as a framework to discuss² logic in terms of practices. In Section 5, I will present an incompatibility semantics based on linear logic and I will show how it is capable of expressing intuitionistic and non-monotonic inferences, by means of a refinement of the notion of defeasibility. Section 6 contains some conclusive remarks on the relationship between inferential practices and the proposed incompatibility semantics.

2 Analytical pragmatism

Without entering the details of the complex theory proposed by Brandom, I briefly recall the definition required in order to state the place of logic in Brandom’s analytical pragmatism. Brandom defines some fundamental *meaning-use* relations that allow to keep track of the pragmatic aspects of meaning definitions.

Consider a set of practice or abilities P and a vocabulary V , one defines two fundamental meaning-use relations: *PV-sufficiency*, which holds between P and V when P is sufficient to deploy V , and *VP-sufficiency*, which holds when the vocabulary V is sufficient to specify the set of practice P .

Using the two fundamental relations, one can define *PP-sufficiency*, that holds between two sets of practice P and P' when the first can be elaborated into the other (by a set of algorithmic abilities that implement practical elaboration), and *VV-sufficiency*, that holds between two vocabularies V and V' when V is sufficient to characterize V' .

An interesting example Brandom presents to interpret those relations in a precise way is given by formal language theory, taking vocabularies in their syntactic aspect. Considering Chomsky hierarchy, classical results in formal language theory show that

¹ [16], §202.

² The point of view here adopted is closer to Brandom foundational considerations. A modelization of speech acts that keep track of their illocutionary force using linear logic has been developed in [3].

context free grammar can be generated by push down automaton and context sensitive language can be generated by a linear bounded automaton.

In this example, the vocabulary V is the language generated by the automaton and the set of practice or abilities P is represented by the computational power of the automaton. For example, push down automata are PV-sufficient for a context free language.

The automata example, allows to state also in a quite precise way the relation of PP-sufficiency as algorithmic elaboration: a set of practice or abilities P is PP-sufficient for P' whenever an algorithm implementing P can also implement P' .

It is interesting to remark that this approach introduces complexity issues concerning the practices we are describing, besides providing a general framework to speak about practice. Consideration concerning complexity are interesting from a cognitive point of view, since they provides useful tests for formal models; for example, one could assume that the tasks actual practices or abilities perform should be described as tractable problems, or problems not exceeding the complexity class NP.

One can also consider meaning-use relations defined in terms of necessity. For example, *VV-necessity* captures the notion of semantic presupposition or, considering automata example, a notion of syntactic presupposition.

PP-necessity holds between P and P' when it is not possible to engage in a set of practice P unless one engage also in the set of practice P' . The notion of PP-necessity plays an important role in Brandom's approach since it allows to formulate in a general and elegant way several kinds of pragmatist arguments, as Sellar's critique of phenomenalist form of empiricism ([5], p.12) and, more generally, Brandom's argument to justify the fundamental role of *inferring* in discursive practices. Let's focus on the notion of *PV-necessity*, stated by Brandom as follows: "the capacity to say something of a certain kind, to deploy a particular vocabulary, can require being able to do something of a specifiable kind" (p. 40).

Using this notion, we can briefly present the fundamental claim which characterizes Brandom's pragmatic rationalism, namely the universal PV-necessity of inferential practice.

Considering discursive practices in general, Brandom claims that *asserting* is a fundamental practice which is necessary in order to engage in any practice we count as discursive. But asserting cannot be considered independently from *inferring*, since assertions are constitutively speech acts that can be used as premises or as conclusions of inferences: asserting and inferring are practice such that each is PP-necessary for the other. Following this point of view, inferential practice are PP-necessary for every practice that may count as discursive, therefore they are necessary to deploy any kind of vocabulary: they are *universally* PV-necessary.

Any kind of practice we consider discursive, any kind of *language* game, must include practices of giving and asking for reasons, since asserting and inferring are deeply connected. Therefore, there is something common to all language games, in Brandom's terms: "pragmatic rationalism is the view that language *does* have a 'downtown', and it comprises the practice of making claims and giving and asking for reasons for them" ([5], p.43.).

We can now present the role of logic in Brandom's approach. A *language of conditionals* can be viewed as the first step leading from the practice of inferring, namely drawing certain pre-logical inferences, towards the effective employment of a logical vocabulary: "for conditionals let one say something, while before one could only do something" (p. 47). In particular, "the expressive role distinctive of conditionals [...]"

is to codify inferences, to specify inferential practice-or-abilities, to explicate them, in the sense of making explicit something that was implicit in them” ([5], p. 47). So the relationship between conditionals and inferential practice can be stated in terms of elaboration and explication: conditionals are *elaborated from* inferential practice and are *explicative of* inferential practice. This is a peculiar type of meaning-use relation, labeled *LX-relation* (*elaborated from* and *explicative of*): the genus of logical vocabulary may be defined precisely as this particular kind of meaning-use relation.

It is useful to recall the point of view on logic adopted in Brandom’s approach: the nature of logical vocabulary is investigated in order to legitimate the use of logic to express considerations about meanings, to state the meaning relations between different vocabularies. Brandom’s arguments are advanced in order to vindicate “semantic logicist commitment in the classical project of analysis”(p.51), so to justify the use of logic in the analysis of meaning. I will not discuss whether this argument justifies the use of logical formalism to represent meanings, I believe that this is a powerful intuition about logic in general, since it leads towards a very interesting approach to foundation of logic which deeply links logic to discursive practice³.

As Brandom summarizes (p. 136), we can see how logical vocabularies are related to the practice of giving and asking for reasons as follows.

Starting from the practice of giving and asking for reasons, Brandom argues, one shows that it is *PP-sufficient* for deploying basic normative vocabulary, in particular the deontic modal vocabulary of *commitment* and *entitlement*; then one may use the modal vocabulary as a pragmatic metavocabulary that specifies how to deploy the concept of *incompatibility*, which is interpreted as constitutively modal notion. Then one can use incompatibility as semantic metavocabulary to define a consequence relation of incompatibility-entailment. The relation incompatibility-entailment is then sufficient to define logical vocabularies.

3 Limits of incompatibility semantics

If we take a closer look at the formal theory Brandom develops, we see that it is committed with the assumption that classical logic, at propositional level, is *the* logic of incompatibility: “we have seen that any standard incompatibility relation has a logic whose non-modal vocabulary behaves classically” ([5], p. 139). Moreover, it turns out in general that all the inferences the notion of incompatibility can express or justify are those that can be explicated by means of a classical consequence relation. The reason is that incompatibility relations can define only “standard” consequence relations, where a standard consequence relation is defined by two properties: *general transitivity* and *defeasibility*.

Consider intuitionistic consequence relation. The first condition recalls cut rule in sequent calculus, and it is of course satisfied by intuitionistic logic, which also satisfies cut elimination. This is not the case for defeasibility, which states intuitively that if a proposition *B* is not a consequence of a proposition *A*, then there is something that yields an absurdity, when added to *B* but not when added to *A*.

³ It would be interesting to take a closer look at the relationship between logic discursive practice and dialogic approach to logic, in particular to compare Brandom’s view with dialogic tradition of Lorenzen [12] and with the recent developments in game semantics. In Section 5, I will take some insights inspired by the dialogic tradition to provide an interactive account of incompatibility detection.

The reason why intuitionistic logic doesn't satisfy defeasibility is that defeasible reasoning demands a *witness* also for the badness of an inference. In intuitionistic logic, a witness of good inferences is always provided in a natural way: an inference from A to B is good if and only if there is a proof of B given A . So we can take the proof to be the witness. Hence in intuitionistic logic there is a strong connection between valid inferences and witnesses⁴.

Consider the case in which B doesn't follow from A . That means that the conditional $A \rightarrow B$ is not true, which entails in intuitionistic semantics that there is no proof of B given A . So in general, the fact that B doesn't follow from A in intuitionistic logic means that there is *no witness*, no proof, of B given A . This is the constructive, or epistemic, character of intuitionistic logic: it shows that we don't have good reasons for what we don't know.

Since intuitionistic inferences cannot be represented by a consequence relation defined by incompatibility relations, intuitionistic logical vocabulary cannot be justified in Brandom's approach. This entails that the analysis of the practice of inferring, based on incompatibility detection is missing something: if the vocabulary of intuitionistic logic cannot be defined in terms of incompatibility relations, then the inferential practice described by Brandom is not universally PV-necessary, since it is not necessary for deploying intuitionistic vocabulary⁵. Assuming classical logic as the vocabulary related to inferential practice, we are implicitly assuming that pre-logical inferences represented by conditionals which differs from classical material implication are in some sense derived from the classical one. Even if non-classical inferences could cleverly be explicated by means of some complicated modal logic construction, their *use* would not be grounded in any inferential practice defined by Brandom. If one consider what we may call *intuitionistic practice of inferring*, according to which we reject for example arguments by contradiction or we require examples to accept existential statements, the only way we have to explicate those inferential practice is by saying that they don't behave classically and then we need to find reasons for this divergence.

Since, as Brandom proves, standard consequence relations are precisely those that can be obtained by means of incompatibility relations ([5] p. 138) and no incompatibility relation can define a non-standard consequence relation, we are lead to admit that we can justify just those inferential practices which behave as classical logic.

This is a serious drawback. Firstly, because classical logic has been widely considered not adequate to model agents reasoning. Furthermore, it turns out that many pre-logical inferences that are actually performed in communication cannot be considered as inferential practice in Brandom's sense, since they cannot be obtained from asserting and inferring. If we assume that there is an intuitionistic inferential practice, besides a classical one, then inferring in Brandom's sense is not PP-necessary anymore.

Consider another example. Assuming standard consequence relation, there is no way to justify causal inferences in terms of discursive practice. Let's consider a toy

⁴ In particular, Curry-Howard isomorphism between proofs in intuitionistic logic and terms in lambda calculus can be considered as an effective construction of witnesses for good inferences.

⁵ The point is that the *relationship* between classical and intuitionistic reasoning cannot be stated in terms of pragmatically mediated semantic relations. I claim that this relationship has a special interest for semantics since, as Dummett points out in [6], classical and intuitionistic logic lead to two different theories of meaning: the first one states meaning in terms of truth conditions, and in general is committed with realism, while the second one gives a characterization of meaning in terms of proof, or reasons, and can be considered matching anti-realist insights.

example of causality. Assuming the notion of incompatibility Brandom axiomatizes, one can prove the following (see [5], p. 128):

$$\text{If } A \text{ entails } B \text{ and } A \text{ entails } C, \text{ then } A \text{ entails } B \text{ and } C. \quad (1)$$

A famous example proposed by Girard in order to explicate the meaning of linear logic connectives is based on inferences like (1). As an example of (1) we can consider: “if I spend 1 euro, I get a coffee”, “If I spend 1 euro, I get a tee”, hence “if I spend 1 euro, I get a coffee and a tee”.

As Girard argues, assuming (1) amount to forget any causal relation between premises and conclusions. The reason is that, briefly, interpreting propositions as events, we do not make distinction between a single occurrence of an event (“spending 1 euro”) and any number of occurrences of an event. In general, since the consequence relation Brandom defines satisfies weakening (see p. 143), non-monotonic reasoning cannot be described as well.

Therefore, if we consider a vocabulary of linear conditional, or a vocabulary of non-monotonic conditionals, we see that they cannot be elaborated from the inferential practice defined by Brandom: this would provide another argument against universal PV-necessity of inferring described in terms of incompatibility detection.

Again, if we consider a practice of making pre-logical causal inferences like (1), the practice of inferring described by Brandom would not be PP-necessary anymore.

Since the notion of incompatibility is not suitable to represent consequence relations which are well codified as intuitionistic and causal reasoning, I claim that the notion of incompatibility as stated by means of general transitivity and defeasibility is not adequate to ground logical vocabularies and to connect logic with inferential and discursive practice. A more fine-grained articulation of the concept of inferential practice is then required in order to place different reasoning patterns under a same pragmatic foundational concept. In the next sections I will show how a formal analysis of discursive practices may also justify a different kind of incompatibility semantics, which is capable of defining intuitionistic, classical and linear (non-monotonic) inference. Before presenting incompatibility semantics, we need to introduce some features of linear logic which is the general framework for the semantics I am proposing.

4 Linear logic

Linear logic is a resource sensitive logic that allows to specify precisely how hypotheses are used in the deduction. Its applications were successful in computer science, providing models of computation, and in general in modelling processes in which awareness of resources consumption is fundamental⁶. We will discuss linear logic and its relationship with classical and intuitionistic logic taking sequent calculus as a model of proof system, since it is particularly apt to speak about properties of proofs.

Consider the behavior of the implication in linear logic. Linear implication, denoted $A \multimap B$, satisfies *modus ponens*:

⁶ An interesting aspect of linear logic is that many abstract machines can be encoded in linear logic (see [8], p. 220-221). We cannot enter the details here, I simply remark that this feature provides a precise definition of Brandom’s idea of LX-relation: if the practice or abilities are represented, as in the formal language example, by means of automata, we can see how to define the notion of *explication* straightforwardly: we simply say that a vocabulary V explicates the practice P if the vocabulary is expressive enough to encode the machine implementing P .

$$A, A \multimap B \vdash B \quad (2)$$

However, (2) is provable just in case the right quantity of antecedent formulas is provided:

$$A, A, A \multimap B \not\vdash B \quad (3)$$

The sequent (3) is not provable since intuitively there is an A which is not demanded by the antecedent of the implication. The meaning of $A \multimap B$ is “consuming A , one can produce B ”. We can see (1) as a form of deal between a buyer and a seller: “I give you A , if you give me B ”. In order to achieve the control on the demands of formulas in the deduction, structural rules of Gentzen sequent calculus (weakening and contraction) do not hold at a global level:

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{W} \quad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{C}$$

Recall that in sequent calculus there are two ways of presenting logical rules: an additive form, which take the union of contexts, and a multiplicative form, which take copies of contexts:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \text{additive} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \text{multiplicative}$$

Using structural rules, one can prove that the two formulations are equivalent. If structural rules are removed, we have to consider two different types of conjunction (and, by duality, two types of disjunction):

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \& \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes$$

The language of linear logic can be divided into three groups of connectives: multiplicative, additive and exponentials. I briefly recall their intuitive meaning in the following table. We will consider negation, the most important linear connective, in more details later.

Table 1. *Linear logic connectives*

<p><i>Multiplicatives:</i> Conjunction: $A \otimes B$ (“tensor”). It means that we have exactly one copy of A and one copy of B, no more no less. E.g. $A \otimes B \not\vdash A$: intuitively, in order to sell A and B, we need someone who agrees to buy A and B, while here there is just a buyer for A. Disjunction: $A \wp B$ (“par”). Its intuitive meaning is clearer considering that it is the disjunction that defines linear implication: $A \multimap B \Leftrightarrow A^\perp \wp B$, where A^\perp is the linear negation of A. Units: $\mathbf{1}$ is the neutral element for \otimes and \perp for \wp. <i>Additives:</i> Conjunction: $A \& B$ (“with”). It introduces a form of choice: we have one between A and B and we can chose which one, for example $A \& B \vdash A$, but we don’t have them both: $A \& B \not\vdash A \otimes B$. Disjunction: $A \oplus B$ (“plus”). It means that we have one between A and B but we cannot chose, $A \vdash A \oplus B$ but $A \oplus B \not\vdash A \& B$. Units: \top is the neutral element for $\&$ and 0 is the neutral element for \oplus. <i>Exponentials:</i> They are unary connectives, denoted $!A$ and $?A$. They allow to reintroduce structural rules in a local way: $!$-formulas allow contraction and weakening on the premises of the sequent (on the left of \vdash), $?$-formulas allow contraction and weakening on conclusions (on the right of \vdash). Exponentials remove formulas from their linear status and make quantity not to matter any more.</p>
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4.1 Translating classical and intuitionistic logic into linear logic

Rather than considering linear logic as an alternative logic, one can see it as a proof-theoretical analysis of classical and intuitionistic logic. The reason is that we can take classical and intuitionistic sequents and translate them in linear logic which provides a framework in which intuitionistic and classical proofs coexist and interact. The idea is that, instead of changing logic or deductive system, one can change formula: one can define intuitionistic or classical formula as linear formula of a certain type. Therefore, in this approach, it is the syntax of the formula to show the deductive properties required to argue on the content defined by that proposition: the formula itself contains the indication showing the kind of reasoning required.

One of the most important remarks that lead to linear logic was the decomposition of intuitionistic implication (\rightarrow) into linear implication and exponential:

$$A \rightarrow B = !A \multimap B \quad (4)$$

Intuitively, it means that intuitionistic implication can be obtained from linear implication forgetting the quantity of hypothesis.

I briefly recall the translation (see [10]) of classical and intuitionistic logic into linear logic since I will use it in order to provide a more general incompatibility semantics to account for classical, intuitionistic and linear inferences.

As it is well known from Gentzen, intuitionistic sequent calculus can be obtained from classical sequent calculus with the condition that the right side of the sequent must contain at most one formula; that is enough to block for example the provability of the *excluded middle*. Intuitionistic linear logic (ILL) is obtained as well from classical linear logic (CLL) restricting the right side of the sequent.

The analysis linear logic provides of classical and intuitionistic proofs is stated in terms of a controlled use of structural rules, so the difference between intuitionistic, classical and linear proofs can be presented in terms of formulas allowing weakening and contraction on the left or on the right of the sequent.

We will not enter the details of the translation here, for all the definitions, I refer to [10]. For example, in the intuitionistic case, one translate atoms $p^* = p$, conjunctions $(A \wedge B)^* = A^* \& B^*$, and implications as $(A \rightarrow B)^* = !(A^*) \multimap B^*$; for the classical case the situation is more complicated since there are choice to be made in order to keep track of the position of the classical formula in the sequent (premise or conclusion).

Let A^* be a translation of intuitionistic (classical) formula in linear logic, we have:

$$\Gamma \vdash_{\text{IL}} A \text{ iff } !\Gamma^* \vdash_{\text{LL}} A^* \quad (5)$$

$$\Gamma \vdash_{\text{CL}} \Delta \text{ iff } !\Gamma^* \vdash_{\text{LL}} ?\Delta^* \quad (6)$$

Intuitively, intuitionistic proofs may use structural rules just on the left of the sequent, being the right side restricted to a single formula; classical proofs may use structural rules on both sides.

We will exploit this translation in order to discuss defeasibility and to present a notion of incompatibility that is capable to account both for defeasible inferences and non-defeasible inferences in an harmonious way.

It is important to remark that the relationship between classical linear logic and intuitionistic linear logic differs from the relationship between classical and intuitionistic logic. In linear logic, if one restricts the language excluding the constant $\mathbf{0}$ and

\perp , then a formula is provable in intuitionistic linear logic if and only if it is provable in classical linear logic [14]⁷. In the following section, we will use intuitionistic linear logic augmented with constant $\mathbf{0}$ and \perp , as in [15].

4.2 Negations

Before presenting an incompatibility semantics inspired by linear logic, we need to consider the role of linear negation. The intuition is that linear negation A^\perp operates a form of *exchange of perspective*, rather than switching the semantic value of a proposition.

In particular, the role of negation can be interpreted as the exchange of perspective between a proponent and an opponent in a dialog game⁸.

The idea is that a certain content “ A ” can be interpreted as a set of abstract objects, they can be considered as actions, or players, in favor of A ; then linear negation A^\perp will be a set of actions, or players challenging A . Then, $(A^\perp)^\perp$ represents the opponent of the opponent of A . Since negation in linear logic is involutive, we have:

$$A^{\perp\perp} = A \quad (7)$$

So an opponent of an opponent of A is a proponent of A , and not just an opponent of A^\perp . I will refine this interpretation in the following section.

Besides linear negation⁹, it is possible to define a usual notion of negation by means of absurdity and implication, as usual in intuitionistic logic, exploiting the intuition: “*not A* means that claiming A , one would claim absurdity”.

Firstly, we remark that in linear logic there are two constants for absurdity: \perp and $\mathbf{0}$. The difference can be presented as follows: \perp doesn’t satisfy *ex falso quodlibet*: even if \perp is derivable, that doesn’t entail that for every formula ϕ , $\perp \vdash \phi$; while $\mathbf{0}$ satisfies *ex falso quodlibet*, being $\Gamma, \mathbf{0} \vdash \Delta$ an axiom of linear logic. So we can define two negations:

$$\neg A \Leftrightarrow A \multimap \mathbf{0} \quad (8)$$

$$\neg_{lin} A \Leftrightarrow A \multimap \perp \quad (9)$$

We have that $A, A \multimap \mathbf{0} \vdash \phi$ for all ϕ , while there are ϕ such that $A, A \multimap \perp \not\vdash \phi$. We will use this negation in Section 5 to provide a more fine-grained analysis of defeasible inferences.

⁷ This property doesn’t hold in the non-linear case. For example, Peirce law $((A \multimap B) \multimap A) \multimap A$, which does not contain the constant for absurd, it is provable in classical logic while it is not provable in intuitionistic logic.

⁸ The dialogical interpretation of logic and in particular the interpretation of sequent calculus in terms of dialog games goes back to the work of Lorenzen ([12]). A dialogic interpretation of linear logic is provided by Blass, see [4]. Further refinements of Blass semantics lead to the definition of a game semantics for linear logic in [1]. Game semantics can be considered a truly interactive account of the meaning of logical constants, which seems to be closer to Brandom’s point of view. Here, we decided to work with the algebraic semantics of linear logic in order to make the comparison with Brandom’s incompatibility semantics easier.

⁹ We can consider linear negation as a type of negation at least considering that it defines De Morgan dualities between conjunction and disjunction: $(A \otimes B)^\perp$ iff $(A^\perp \wp B^\perp)$.

5 Incompatibility Semantics based on linear logic

I present an interpretation of linear logic semantics in terms of reasons used in an abstract dialog. In [7], Girard suggested an intuitive interpretation of phase semantics, the algebraic semantic providing a canonical model of linear logic, in terms of *phases* of observations and *facts*. The analogy was inspired by quantum mechanics: a *fact* F , which is in linear logic the semantic value of sentences, is a set of observations for which a form of agreement between observer and observed holds. This form of agreement is represented by the property $F = F^{\perp\perp}$, so facts are those sets of observations that are stable under the test represented by the counter-observations F^\perp .

I suggest here a different interpretation based on what I call *actions that count as reasons* and *propositions*. The notion of action is here intended to be very abstract, no further specification is provided, in particular we do not require any normative constraint on what kind of actions a count as a reason for an atomic content A , just the agreement on the fact that a count as a reason. So we are not going to define good or bad reasons for atomic contents, rather we just consider admissible reasons.

The intuition motivating this interpretation is that the practice of giving and asking for reasons requires a form of agreement between what counts as a reason for accepting A and what count as a reason for rejecting A . In our framework, the practice of giving and asking for reasons requires that a proponent must be recognized as a proponent and an opponent must be recognized as an opponent on some issue. This form of agreement has been analyzed by Brandom in terms of commitment and entitlement. I will argue that this form of agreement can be formalized by means the properties of linear negation $A = A^{\perp\perp}$.

Brandom's approach concerning incompatibility semantics can be summarized as follows: firstly, the analysis of discursive practices as practices of giving and asking for reasons by means of the concepts of commitment and entitlement leads to define incompatibility relations; then, exploiting the properties of incompatibility relations, one defines a consequence relations; then one proves representation theorems stating that incompatibility relations generates precisely those consequence relations.

Here, since the notion of incompatibility I define is based on the semantics of linear logic¹⁰, we will have representation theorems respect linear consequence relation for free. Moreover we have soundness and completeness respect sequent calculus for linear logic.

5.1 Actions that count as reasons

Let $(P, \cdot, 1)$ be a commutative monoid, the intended meaning of the elements of the monoid is that they are actions that count as reasons in particular contexts. The multiplication \cdot represents a concatenation of such actions, one may imagine it as the sequence of moves in a dialog. Therefore actions do not behave as elements of sets, since their repetitions matters¹¹. The unit 1 of the monoid is a special action that changes

¹⁰ The definitions are just an interpretation of those defining phase semantics for linear logic, see [8]. We are going to define a *classical* phase space which provides models for classical linear logic; then we will point at some differences with intuitionistic linear logic discussing defeasible inferences.

¹¹ The order of actions should matter too, we assume here commutativity for simplicity. Note that it is possible to consider non-commutative versions of linear logic; for an algebraic semantics, see [17].

nothing: given any action a , performing 1 one has a : $a \cdot 1 = a$. The unit 1 will play the role of a special action that count as reason: if 1 is reason for A , then the agreement on the issue A has been reached. The unit 1 will be used to define the notion of truth in this context.

We define on subsets $X, Y \subseteq P$, the following operation:

$$X \multimap Y := \{a \mid \forall x \in X, ax \in Y\} \quad (10)$$

$X \multimap Y$ is the set of those actions a that concatenated with an action x in X , produce an action in Y . A *phase space* is defined as follows:

Definition 1 A *Phase space* \mathcal{P} is a couple (\mathbf{P}, \perp) where \mathbf{P} is a commutative monoid and $\perp \subseteq P$ is a chosen subset of P called the *pole*.

In our interpretation, the choice of the pole \perp amounts to assume a form of agreement between agents on what count as *disagreement*: the choice of the pole can be seen as the choice of those actions that are considered *incompatible*.

Given a subset X of P , we define the following operation that interpret linear negation:

$$X^\perp := X \multimap \perp = \{y \in P \mid \forall x \in X, yx \in \perp\} \quad (11)$$

The set X^\perp is the set of those actions that are incompatible with actions in X . Intuitively, negation allows to define directions: if A can be considered a set of reason *for*, then A^\perp can be considered a set of reason *against*.

We can now define an incompatibility relation between elements of the phase space \mathcal{P} .

Definition 2 An *incompatibility relation*, $a \perp b$, between a and b in P holds iff $ab \in \perp$

We are defining the incompatibility relation between actions in general. Once we define propositional contents, we will have a definition of incompatibility between reasons. The sets of elements of \mathcal{P} that provide denotation of sentences, namely those sets of actions providing propositional contents, are defined as follows.

Definition 3 A *proposition* is a subset $A \subseteq P$ such that $A = A^{\perp\perp}$. The elements of a proposition A are called *reasons*.

In the interpretation I am suggesting, A is a set of actions a that may count as reason for a particular content; A^\perp will be the set of those actions a' such that $aa' \in \perp$, therefore a' are those actions that together with an a for A produce an action of the type \perp , namely a and a' are incompatible. The negation $()^\perp$ operate the exchange between proponent and opponent.

We can imagine this abstract communicative situation as follows: a proponent P performs an action a to claim an issue A , then the opponent O challenges a by means of a' which is in A^\perp ; then P replies with something in $A^{\perp\perp}$ and so on. Since $A^{\perp\perp}$ is equal to A , the form of agreement we are assuming means that P and O are aware they are discussing a *same* issue¹².

¹² We could say that it is the property itself that let us speak of a *same* issue, namely, an issue is determined by the interaction in the dialog.

$$\underbrace{a : A}_P, \text{ then } \underbrace{a' : A^\perp}_O, \text{ then } \underbrace{a'' : (A^\perp)^\perp}_P \dots \quad (12)$$

Not every set of actions X in P has this property. Consider what happens on those sets of actions that are not propositions, namely, those sets X such that $X \neq X^{\perp\perp}$. Since for every subset X it holds that $X \subseteq X^{\perp\perp}$, if X is not a proposition, it means that there are actions for $X^{\perp\perp}$ that are not actions for X : a move x'' to challenge a move x' for X^\perp (which was challenging $x : X$) is not recognized as a reason for X , but for some other content, on which there is no agreement on what actions are considered incompatible:

$$\underbrace{x : X}_P, \text{ then } \underbrace{x' : X^\perp}_O, \text{ then } \underbrace{x'' : (X^\perp)^\perp = Y}_P \text{ then } \underbrace{x''' : ((X^\perp)^\perp)^\perp = (Y)^\perp}_O \quad (13)$$

In (13), x and x' are incompatible by definition, but then, there is no agreement on the fact that x'' and x''' are incompatible. So there is no agreement on the fact that x''' is challenging x'' , and hence no agreement on the fact that they count as reasons for the agents involved in the communication¹³.

Among the properties that holds in a phase space, we have that for any subset $X \subseteq P$, the smallest proposition containing X is given by $X^{\perp\perp}$. Moreover, every proposition A is of the form Y^\perp . Intuitively, a propositional content A requires agreement on which set of actions (Y) are those that challenge the content of A . It is important to remark that the notion of incompatibility we defined here depends on the choice of the pole. As we saw, the notion of incompatibility is defined in terms of \perp , so there may be different form of incompatibility depending on the context, on the choice of the specific subset \perp .

Since \perp is a proposition (it satisfies $\perp^{\perp\perp} = \perp$), the model requires that agents agree on what counts as a reason for \perp : it is required to agree on what count as disagreement, on what actions are incompatible. Moreover, there may be different types of *propositions*, or propositional contents, that depend on the choice of what is disagreement. For example, if one takes the pole \perp to be empty, then we have just two propositions P and \emptyset , or two kinds of semantic value, so we have the usual truth values semantics for classical logic. In that case disagreement is just classical contradiction.

5.2 Logical connectives

We are going to define logical connectives on propositions. Remark that the structure we presented is quite explicit on which type of contents may be object of logic: logic works on propositions, that are those contents on which there is a agreement on what

¹³ Remark that the technical framework I am presenting can be considered independently from the suggested interpretation. The interpretation here defined points at a purely pragmatic interpretation of propositional content of sentences, which is identified with sets of actions that count as reasons in a dialog. One could anyway take the technical definitions and state them as in [5], see technical appendix p. 141-155. Define an *incompatibility frame* as a *syntactic phase space* (\mathcal{P}, \perp) , where P is a set of propositional formulas; example of syntactic phase spaces are involved in the completeness proof of sequent calculus for linear logic, see [8]. The incompatibility-entailment relation can be defined as $X \models_\perp Y$ iff $X^\perp \subseteq Y$. In this way \models_\perp will define precisely the linear logic consequence relation.

count as a reason.¹⁴ The following properties are used to define the interpretation of connectives:

Proposition 1 For every $A, B \subseteq \mathcal{A}$:

1. if $A \subseteq B$, then $B^\perp \subseteq A^\perp$
2. $A \subseteq A^{\perp\perp}$
3. $A^\perp = A^{\perp\perp\perp}$
4. A is a proposition iff $A = B^\perp$, for some $B \in \mathcal{A}$
5. $(A \cup B)^\perp = A^\perp \cap B^\perp$

Define the concatenation $X \cdot Y$ of sets as follows:

$$X \cdot Y = \{xy \mid x \in X \text{ and } y \in Y\} \quad (14)$$

We can now define the semantic structures required to interpret linear connectives. First one defines linear negation using \perp :

$$A^\perp = (A)^\perp \quad (15)$$

For every proposition $A, B \subseteq \mathcal{P}$, we have:

$$\begin{array}{llll} A \otimes B & := (A \cdot B)^{\perp\perp} & A \& B & := A \cap B \\ A \wp B & := (A \cdot B)^\perp & A \oplus B & := (A \cup B)^{\perp\perp} & !A & := (A \cap I)^{\perp\perp} \\ \mathbf{1} & := \{1\}^{\perp\perp} & \mathbf{0} & := \emptyset^{\perp\perp} & ?A & := (B \cap I)^{\perp\perp} \\ & & \top & := \mathcal{P} & & \end{array}$$

In the definition of exponentials, I is the set of the idempotent elements of the monoid \mathbf{P} , namely those elements that satisfy $a \cdot a = a$, intuitively they are those actions for which repetitions doesn't matter. Note that there is at least one idempotent defined by the unit of the monoid.

For reasons of space, we can only mention that it is possible to provide a form of dialogic interpretation of linear logic connectives, namely it is possible to define them as moves in a dialogue game, we refer to [4].

5.3 Linear consequence relation

The usual semantic notions of truth, consequence and validity can be stated in this framework as follows.

Let \mathcal{L}_{LL} be a language for linear logic:

$$\mathcal{L}_{LL} := p \mid p^\perp \mid A \otimes B \mid A \wp B \mid A \oplus B \mid A \& B \mid !A \mid ?A \quad (16)$$

We will use Greek capital letters to denote (multi) sets of formula¹⁵.

¹⁴ It would be interesting to consider the relationship between logic and other form of discursive dynamics: if logic works where the agreement we defined holds, how communication can be formalized when that form of agreement is missing? Is it still a form of reasoning? Interesting insights on this issue can be found in Girard's *Ludics*, a recent development of linear logic, see [9]. The connection between the paradigm of computation defined in Ludics and Wittgenstein's language games has been investigated in [13].

¹⁵ In classical logic one consider sequents to be defined as sets of formulas. In linear logic, we have to consider multi-sets since repetitions matter.

Let \mathcal{P} be a phase space, the interpretation of \mathcal{L}_{LL} into \mathcal{P} is a function v that associates with each atomic formula p in \mathcal{L}_{LL} a proposition $v(p) \subseteq \mathcal{P}$. We can extend v to complex formulas using Proposition 1.

The notion of truth in this context is defined by means of the neutral element of the monoid underlying \mathcal{P} :

Definition 4 A formula A is *true in \mathcal{P}* iff $1 \in v(A)$. A formula A is *valid* iff A is true in every phase space \mathcal{P} .

A proposition holds in a phase structure when 1 belongs to the interpretation of A . The notion of validity states that A is true *for every* phase space: in particular, it means that A is true for every choice of \perp . Valid formulas are those that are true independently of the choice of what count as incompatible.

We can define a consequence relation for linear logic, denoted by \models_{LL} , as follows:

Definition 5 $\Gamma \models_{LL} \Delta$ iff for every valuation v , $v(\Gamma) \subseteq v(\Delta)$.

So valid inferences are those that do not depend on what count as incompatible, they hold for every choice of \perp .

We have soundness and completeness of linear logic sequent calculus respect to phase semantics:

Theorem 1 $\Gamma \models_{LL} \Delta$ iff $\Gamma \vdash_{LL} \Delta$.

5.4 Defeasible inferences

In this section, I present a refinement of the notion of defeasibility in order to account for the non-monotonic and causal inferences that can be expressed in linear logic. We are going to prove that linear consequence relation will satisfy general transitivity at a global level, while it will satisfy defeasibility just in a local way, namely on particular types of propositions. Then we will see which kind of formula are those involved in defeasible inferences. This will provide a more fine-grained analysis of types of inference in terms of what kind of reasons they demand.

We can rephrase Brandom's definitions in our framework as follows:

Definition 6 A consequence relation \vdash is *standard* iff it satisfies¹⁶:

1. *General transitivity* (GT): if $\Gamma \vdash_{LL} A$ and $\Delta, A \vdash_{LL} B$, then $\Gamma, \Delta \vdash_{LL} B$.
2. *Defeasibility* (D): Given Γ and Δ , if $\Gamma \not\vdash_{LL} \Delta$, then there exists Σ such that, for every B , one has $\Delta, \Sigma \vdash_{LL} B$ and there exist a formula C such that $\Gamma, \Sigma \not\vdash_{LL} C$.

In linear logic, we can refine the notion of defeasibility considering two formulations:

Definition 7 Depending on the choice of absurdity, we have the following notions of defeasibility:

1. (D_0) : if $\Gamma \not\vdash_{LL} \Delta$, then there exists Σ such that $\Delta, \Sigma \vdash_{LL} 0$ and $\Gamma, \Sigma \not\vdash_{LL} 0$.

¹⁶ I state the definition using the syntactic notion of provability since we will use it to prove the results. By completeness theorem, the same holds for semantic consequence relation.

2. (D_{\perp}) : if $\Gamma \not\vdash_{LL} \Delta$, then there exists Σ such that $\Delta, \Sigma \vdash_{LL} \perp$ and $\Gamma, \Sigma \not\vdash_{LL} \perp$.

(D_0) is equivalent to defeasibility as defined by Brandom, when one considers classical or intuitionistic formula embedded in linear logic. Remark that it is the analysis provided by linear logic that allows to see the difference of the two formulations which entails an interesting classification of inferences¹⁷.

Proposition 2 *Let \vdash_{CL} be the translation of classical logic in linear logic, \vdash_{IL} the translation of intuitionistic logic, \vdash_{LL} full linear logic (including classical logic) and \vdash_{ILL} intuitionistic linear logic (including intuitionistic logic):*

1. \vdash_{LL} satisfies (GT);
2. \vdash_{LL} satisfies (D_{\perp}) ;
3. \vdash_{LL} doesn't satisfy (D_0) ;
4. \vdash_{ILL} doesn't satisfy (D_{\perp}) nor (D_0) ;
5. \vdash_{CL} satisfies (D_0) and (D_{\perp}) ;
6. \vdash_{IL} doesn't satisfy (D_0) nor (D_{\perp}) .

Proof Remember that CL and IL are obtained restricting LL to particular formulas and sequents. For 5. and 6. there is nothing to prove, since we can consider the translations of classical and intuitionistic sequents in linear logic that conserve provability. In particular, for \vdash_{IL} , we can take the translation of the example used by Brandom ([5], p. 171) $\neg\neg p \not\vdash_{IL} p$, which is $!(!(p \multimap 0) \multimap 0) \not\vdash_{ILL} p$.

The other claims are proved as follows.

1. The case of (GT) easily follows, since \vdash_{LL} enjoys cut elimination. Remark that, by completeness theorem, we have also that \models_{LL} satisfies (GT).
2. \vdash_{LL} satisfies (D_{\perp}) . Let $\Gamma \not\vdash_{LL} \Delta$. We prove that there exists a defeasor Σ . Define Σ as Δ^{\perp} , we have, considering the definition of Δ^{\perp} , that $\Delta, \Delta \multimap \perp \vdash_{LL} \perp$; moreover $\Gamma, \Delta^{\perp} \not\vdash_{LL} \perp$ since, otherwise, from $\Gamma, \Delta^{\perp} \vdash_{LL} \perp$, using the rule of negation, we get $\Gamma \vdash_{LL} \Delta \wp \perp$ which entails, since \perp is neutral for \wp , $\Gamma \vdash_{LL} \Delta$, against the hypothesis. So Δ^{\perp} is a defeasor for $\Gamma \not\vdash_{LL} \Delta$.
3. \vdash_{LL} doesn't satisfy (D_0) . Consider the following sequent which is not provable in LL¹⁸:

$$A, B, C \not\vdash C \tag{17}$$

We prove that, for every Σ such that $\Sigma, C \vdash 0$, we have $\Sigma, A, B, C \vdash 0$. From the hypothesis $C, \Sigma \vdash_{LL} 0$, introducing the implication, we get $\Sigma \vdash_{LL} C \multimap 0$.

Consider the following defeasor of $\{A, B, C\}$, $A \otimes B \otimes C \multimap 0$:

$$A, B, C, A \otimes B \otimes C \multimap 0 \vdash_{LL} 0$$

Using the following provable sequent $C \multimap 0 \vdash_{LL} A \otimes B \otimes C \multimap 0$ ¹⁹ we have that:

¹⁷ The remark concerning two notions of absurdity constitutes a partial answer to the question raised by Brandom on the status of relevant logics, see [5], pp. 173-175.

¹⁸ The reason why (17) is not provable in LL is that it should be obtained from an axiom $C \vdash C$ by means of monotonicity, which doesn't hold at a global level in linear logic.

¹⁹ This sequent can be proved in LL. Intuitively it means that if C proves the *ex falso quodlibet* absurdity 0, then C together with any other formula proves 0. This shows a form of monotonicity implicit in 0.

$$\Sigma \vdash_{\text{LL}} C \multimap 0 \vdash_{\text{LL}} A \otimes B \otimes C \multimap 0$$

by cut, we have:

$$\frac{A, B, C, A \otimes B \otimes C \multimap 0 \vdash_{\text{LL}} 0 \quad \Sigma \vdash_{\text{LL}} A \otimes B \otimes C \multimap 0}{A, B, C, \Sigma \vdash_{\text{LL}} 0} \text{ cut}$$

Considering (17), every Σ that proves 0 with C , proves 0 also with $\{A, B, C\}$, against (D_0) .

4. \vdash_{ILL} doesn't satisfy (D_{\perp}) nor (D_0) . Consider the following sequent:

$$(A \multimap \perp) \multimap \perp \not\vdash_{\text{ILL}} A \quad (18)$$

We prove that every defeasor of A is a defeasor of $(A \multimap \perp) \multimap \perp$. Let Σ be such that $A, \Sigma \vdash_{\text{ILL}} \perp$. Then, we have $\Sigma \vdash_{\text{ILL}} A \multimap \perp$.

Using cut, we have:

$$\frac{(A \multimap \perp) \multimap \perp, A \multimap \perp \vdash_{\text{ILL}} \perp \quad \Sigma \vdash_{\text{ILL}} A \multimap \perp}{(A \multimap \perp) \multimap \perp, \Sigma \vdash_{\text{ILL}} \perp} \text{ cut}$$

So Σ proves \perp also with $(A \multimap \perp) \multimap \perp$. By a similar argument, we can show that ILL doesn't satisfy (D_0) .

□

Proposition 2 shows how defeasibility behaves in the context of classical, intuitionistic and non-monotonic (linear) reasoning. For classical reasoning, the two form of defeasibility are satisfied, moreover one can prove that they are equivalent, since in classical logic the two forms of absurdity collapse.

For intuitionistic logic, both linear and non linear, we have that no form of defeasibility can be satisfied since, as Brandom remarks, intuitionistic logic *lacks* formulas expressing defeasors for bad inferences²⁰.

In linear logic the situation is more complex, since we saw that linear reasoning does not satisfy D_0 but still satisfy D_{\perp} .

Consider a non-monotonic inference like (17). If we look for a witness for the badness of (17) in LL, we can find it in C^{\perp} . Exploiting our definitions, we have that there is agreement on the fact that C and C^{\perp} are incompatible. However $\{A, B, C\}$ and C^{\perp} are not to be considered incompatible, since we lack an explicit agreement on A, B . Intuitively, we can say that in (17) we can find reasons rejecting C , but they are not enough to reject A, B, C : we miss the *relevant* reasons to reject A and B . This is the constructive character of linear logic: in order to challenge a claim, we have to present explicitly our reasons. However, if we look for a *classical witness* of the badness of (17),

²⁰ This treatment of intuitionistic logic is still not fully satisfactory since it is a negative characterization that states that intuitionistic inferences are those that lacks defeasors. However, the approach proposed allow to place intuitionistic logic within a same framework, the one defined by linear logic. So we can state explicitly within the model that intuitionistic inferences lack defeasors, as the formulas restriction shows. We decided here to study the relationship between classical and intuitionistic logic with respect to defeasibility from a syntactic point of view, namely considering provability. We leave to a future work the comparison between classical and intuitionistic phase spaces with respect to incompatibility semantics. Algebraic investigations on the properties of intuitionistic and classical phase spaces are provided in [11].

namely something that entails everything when added to C but not when added to $\{A, B, C\}$, we cannot find it: this notion of incompatibility based on *ex falso quodlibet* forgets the relevance between reasons and claims.

The difference between classical or intuitionistic inferences, which are not resource-sensitive, and linear inference can be discussed considering again our toy example of causality.

$$A \multimap B, A \multimap C \not\vdash_{LL} A \multimap B \otimes C \quad (19)$$

A defeasor for $A \multimap B \otimes C$, which is something that yields \perp with $A \multimap B \otimes C$, has to mention just one copy of A , so it will not produce \perp together with $\{A \multimap B, A \multimap C\}$, for *two* copies of A are required. Intuitively, a reason showing that I cannot get a coffee and a tee with one euro is not a reason showing that I cannot get a coffee with one euro and a tee with another euro.

Consider the intuitionistic counterpart of (19), which is provable:

$$!A \multimap B, !A \multimap C \vdash_{IL} !A \multimap B \otimes C \quad (20)$$

If we were looking for the defeasors we found for (19), we would see that a defeasor for $!A \multimap B \otimes C$ needs to mention *any number of As*, this is the meaning of $!A$. Therefore, it would be a defeasor also for $\{!A \multimap B, !A \multimap C\}$, for to have “any number of A ” twice is to have again “any number” of As . In this way, linear logic provides an analysis of the relevance of reasons, at least allowing to keep track of the *types* of actions that count as reasons required by different reasoning patterns.

6 Conclusions

We saw that linear logic provides an incompatibility semantics which accounts for different types of inferences in terms of types of reasons required. Moreover, we presented a more fine-grained analysis of the notion of defeasibility, in particular retrieving a *relevant defeasibility* which is suitable to characterize non-monotonic inferences. In this way, the proposed incompatibility semantics can account for non-monotonic and resource-sensitive inferences, extending Brandom’s formal approach. We saw that the form of agreement on what counts as incompatible lead us to define a pole \perp and the structure of phase space that provides the semantics of logical connectives. If we describe discursive practices as grounded on the form of agreement on what actions are to be considered incompatible, we can justify linear inferences and so using the translation of classical and intuitionistic logic in linear logic, we can also justify classical and intuitionistic inferences.

Therefore this kind of practice or ability, that could be seen as an abstract negotiation on what counts as incompatible, would be sufficient to deploy linear, classical and intuitionistic vocabularies. Moreover, we have developed the analogy between the agreement on what count as incompatible and an abstract dialogical situation between a proponent and an opponent which are recognized as such, namely whose actions are recognized as reasons. I am not claiming that this kind of practice would be universally PV-necessary, since an analytical account of that practice should require further investigations, for example a comparison with the concepts of commitment and entitlement, and since there might be logical vocabularies that say something important concerning our inferential practice or abilities that are not definable within linear logic.

However, the approach proposed shows at least how a more articulated account of discursive practices is able to justify different kinds of consequence relation that depend on the particular form of agreement practitioners can reach. As a direction for further investigations, I would like to suggest that the (consequence) relations we can elaborate from different forms of agreement might not be restricted to the domain of *logic*, the same process may be used to provide descriptions of more general aspects of language games.

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