Capturing Model-Based Ontology Evolution at the Instance Level: the Case of *DL-Lite*

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Abstract

Evolution of Knowledge Bases (KBs) expressed in Description Logics (DLs) has gained a lot of attention recently. Recent studies of the topic mostly focused on model-based approaches (MBAs), where the evolution of a KB results in a set of models. For KBs expressed in tractable DLs, such as those of the *DL-Lite* family, which we consider here, it was shown that one is faced with inexpressibility of evolution, i.e., the result of evolution of a *DL-Lite* KB in general cannot be expressed in *DL-Lite*. What is still missing in these studies is a thorough *understanding* of various important aspects of the evolution problem for *DL-Lite* KBs: in which fragments can evolution be captured? What causes the inexpressibility? Which logic is sufficient to express evolution? Can one approximate evolution in *DL-Lite*, and if yes, how? This work provides some understanding of these issues for an important class of quite natural MBAs, which cover the case of both update and revision. We describe what causes inexpressibility, and we propose techniques (based on what we call prototypes) that help to approximate evolution according to the well-known Winslett's approach, which is inexpressible in *DL-Lite*. We also identify *DL-Lite* fragments for which evolution is expressible, and for such fragments we provide polynomial-time algorithms to compute the result of evolution.

Keywords: Semantic Web, Description Logics, knowledge base evolution, algorithms, complexity

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1. Introduction

Description Logics (DLs) provide excellent mechanisms for representing structured knowledge. In DLs, a Knowledge Base (KB) consists of two components. The first component of a KB is a TBox, which describes general knowledge about an application domain in terms of classes of objects with common properties, so called *concepts*, and binary relationships between objects, so called *roles*. The second component of a KB is an ABox, which describes facts about individual objects. DLs constitute the foundations for various dialects of the Web Ontology Language (OWL) [20], which is the language standardized by the World Wide Web Consortium (W3C) for representing ontologies in the Semantic Web.

Traditionally DLs have been used for modeling at the intensional level the *static* and structural aspects of an application domain [3]. Recently, however, the scope of KBs has broadened, and they are now used also for providing support in the maintenance and *evolution* phase of information systems. This makes it necessary to study *evolution of KBs* [12], where the goal is to incorporate new knowledge $\mathcal{N}$ into an existing

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KB $\mathcal{K}$ so as to take into account changes that occur in the underlying application domain. In general, $\mathcal{N}$ is a set of formulas that represents properties that should be true after $\mathcal{K}$ has evolved, and the result of evolution, denoted $\mathcal{K} \diamond \mathcal{N}$, is also intended to be a set of formulas. In the case where $\mathcal{N}$ interacts with $\mathcal{K}$ in an undesirable way, e.g., by causing the KB or relevant parts of it to become unsatisfiable, the new knowledge $\mathcal{N}$ cannot be simply added to the KB. Instead, suitable changes need to be made in $\mathcal{K}$ so as to avoid this undesirable interaction, e.g., by deleting parts of $\mathcal{K}$ conflicting with $\mathcal{N}$. Different choices for changes are possible, corresponding to different approaches to the semantics of KB evolution [1, 13, 11].

An important group of approaches to evolution semantics, on which we focus in this paper, is called *model-based*. Under model-based approaches (MBAs) the result of evolution $\mathcal{K} \diamond \mathcal{N}$ is a *set of models* of $\mathcal{N}$ minimally distant from the models of $\mathcal{K}$. Depending on how the distance between models is specified and how it is measured, different MBAs can be defined. In this paper, we refer to eight different MBAs, first introduced in [8], which we will define and discuss in detail in Section 2.2. Since the result of evolution $\mathcal{K} \diamond \mathcal{N}$ is a set of models, while $\mathcal{K}$ and $\mathcal{N}$ are logical theories, it is desirable to represent $\mathcal{K} \diamond \mathcal{N}$ as a logical theory using the same language as for $\mathcal{K}$ and $\mathcal{N}$. Thus, looking for representations of $\mathcal{K} \diamond \mathcal{N}$ is one of the main challenges in studies of evolution under MBAs. When $\mathcal{K}$ and $\mathcal{N}$ are propositional theories, representing $\mathcal{K} \diamond \mathcal{N}$ is well understood [11], while it becomes dramatically more complicated as soon as $\mathcal{K}$ and $\mathcal{N}$ are first-order, e.g., DL KBs [18].

Model-based evolution of KBs where both $\mathcal{K}$ and $\mathcal{N}$ are expressed in a language of the *DL-Lite* family [5] and the evolution semantics is one of the MBAs, has been recently studied [18, 10, 22]. The focus on *DL-Lite* is not surprising since this logic has been specifically designed to capture the fundamental constructs of widely used conceptual modeling formalisms, such as UML Class Diagrams and the Entity-Relationship model [4, 6]. *DL-Lite* is also at the basis of OWL 2 QL [20], one of the tractable fragments (or profiles) that are part of the OWL 2 standard. It has been shown that for each of the eight quite natural model-based semantics introduced in [8], one can find *DL-Lite* KBs $\mathcal{K}$ and $\mathcal{N}$ such that $\mathcal{K} \diamond \mathcal{N}$ cannot be expressed in *DL-Lite* [8, 9], that is, *DL-Lite* is *not closed* under MBA to evolution. This phenomenon was also noted in [18, 8] for some of the eight semantics. What is missing in all these studies of evolution for *DL-Lite* is a thorough *understanding* of

(1) DL-Lite wrt evolution: Which fragments of *DL-Lite* are closed under model-based evolutions? Which *DL-Lite* formulas are responsible for inexpressibility of model-based evolution?

(2) Evolution wrt DL-Lite: Is it possible to capture evolution of *DL-Lite* KBs in richer logics and how can it be done? Which are these logics?

(3) Approximation of evolution results: For *DL-Lite* KBs $\mathcal{K}$ and $\mathcal{N}$, is it possible to obtain "good" approximations of $\mathcal{K} \diamond \mathcal{N}$ in *DL-Lite* and how can it be done?

In this paper we study these problems for evolution that affects the ABox level only, so-called *ABox evolution*, where $\mathcal{N}$ is an ABox and the TBox of $\mathcal{K}$ should stay the same before and after the evolution. ABox evolution is relevant in those settings, where the structural knowledge (TBox) is well crafted and stable, while (ABox) facts about specific individuals may get changed, or new facts may be inserted in the ABox. These ABox changes should be reflected in the resulting KB without affecting the TBox. Our study covers both cases of ABox updates and ABox revision [13]. One significant area where ABox evolution is particularly relevant is Semantic Web services, where one is interested in studying the effects of services that perform operations over the instance data. Such data are inherently incomplete and thus can be effectively represented by means of an ABox. Moreover, they have to obey the semantics of the domain of interest, which is is modelled through a TBox that is assumed not to change over time.

We now describe the contributions that we provide in this paper, and how the paper is organized. In Section 2 we give preliminaries: we review relevant notions from Description Logics and introduce the notion of evolution formally. Our contributions are:

- We introduce the notion of *subsumption* between evolution semantics and discuss in detail for the DL *DL-Lite*$_{\mathcal{FR}}$, a prominent member of the *DL-Lite* family, the subsumption relationship between different MBAs to evolution (Section 3).
- We introduce *DL-Lite*$_{\mathcal{R}}^{pr}$, a restriction on *DL-Lite*$_{\mathcal{R}}$ in which disjointness between concepts and role projections is forbidden. We prove that *DL-Lite*$_{\mathcal{R}}^{pr}$ is closed under most of MBA to evolution and provide two polynomial-time algorithms that compute (representations of) $\mathcal{K} \diamond \mathcal{N}$ for six MBAs (Section 4). For the remaining two MBAs we provide a polynomial-time algorithm to compute maximal sound approximation of evolution.
- For the full DL *DL-Lite*$_{\mathcal{R}}$ we focus on an important MBA corresponding to the well accepted *Winslett's semantics* (WS) [23] (Section 5):
  - We show which combination of TBox and ABox assertions in $\mathcal{K}$ together with ABox assertions of $\mathcal{N}$ leads to inexpressibility of $\mathcal{K} \diamond \mathcal{N}$ in *DL-Lite*$_{\mathcal{R}}$ under WS (Section 5.1).
  - We introduce *prototypes*, which are a generalization of canonical models, and show how they can be used to efficiently approximate $\mathcal{K} \diamond \mathcal{N}$ for WS (Sections 5.2, and 5.3).

Note: A preliminary version of some of this work was presented at two workshops [16, 15] and a conference [14]. New material includes in particular an efficient approximation of evolution under WS, proofs, examples, and insights into why *DL-Lite*$_{\mathcal{R}}$ is not closed under WS. Due to space limitations some proofs are omitted. A version with an appendix that contains additional proofs is available as a technical report at `http://www.inf.unibz.it/~zheleznyakov/techreports.htm` [17]

2. Preliminaries

We introduce now some basic notions about Description Logics (DLs) and present the specific DLs that we consider in this paper. We then define the problem of KB evolution, concentrating on model-based semantics.

2.1. The Description Logic DL-Lite$_{\mathcal{FR}}$

In DLs [3], the domain of interest is modeled by means of *concepts*, denoting sets of objects, and *roles*, denoting binary relations between objects. Complex concepts and roles are obtained from atomic ones by applying suitable constructs. We consider here the logic *DL-Lite*$_{\mathcal{FR}}$, which is a significant member of the *DL-Lite* family of lightweight DLs [5, 2]. In *DL-Lite*$_{\mathcal{FR}}$, (complex) concepts and roles are constructed according to the following syntax

$$B ::= A \mid \exists R, \qquad C ::= B \mid \neg B, \qquad R ::= P \mid P^-,$$

where A denotes an *atomic concept*, B a *basic concept*, and C an arbitrary concept. P denotes an *atomic role* and R an arbitrary role, i.e., either a direct or inverse role.

In *DL-Lite*$_{\mathcal{FR}}$ the knowledge about the domain of interest is represented by means of a *knowledge base* (KB) $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, constituted by a set of assertions partitioned into a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. The TBox consists of concept, role *inclusion assertions* and role *functionality assertions* respectively of the form

$$B \sqsubseteq C, \qquad R_1 \sqsubseteq R_2, \qquad (\mathsf{funct}\ R),$$

which are used to assert intensional domain knowledge. *DL-Lite*$_{\mathcal{R}}$ is a fragment of *DL-Lite*$_{\mathcal{FR}}$ where no functionality assertions are allowed in TBoxes. The ABox consists of *membership assertions* (MAs) of the form

$$P(c, d), \qquad \exists R(c), \qquad A(c), \qquad \neg A(c),$$

where c and d are constants. MAs of the third form are called *negative*, while MAs of the first and second forms are called *positive*.

The *active domain* of $\mathcal{K}$, denoted *adom*$(\mathcal{K})$, is the set of all constants occurring in $\mathcal{K}$. The signature $\Sigma(\phi)$ of an MA ϕ is the set of concept and role names occurring in ϕ, and the signature $\Sigma(\mathcal{K})$ of a KB $\mathcal{K}$ is the set of all concept and role names occurring in $\mathcal{K}$. The size $|\mathcal{K}|$ of a KB $\mathcal{K}$ is the size $|\Sigma(\mathcal{K})|$ of its signature.

The semantics of *DL-Lite*$_{\mathcal{FR}}$ KBs is given in the standard way, using first order interpretations, which we assume to be all over the same countable domain Δ. An *interpretation* $\mathcal{I}$ is a function $\cdot^{\mathcal{I}}$ that assigns to each concept C a subset $C^{\mathcal{I}}$ of Δ, and to each role R a binary relation $R^{\mathcal{I}}$ over Δ in such a way that $(\neg B)^{\mathcal{I}} = \Delta \setminus B^{\mathcal{I}}$, $(\exists R)^{\mathcal{I}} = \{a \mid \exists a'.(a, a') \in R^{\mathcal{I}}\}$, $(P^-)^{\mathcal{I}} = \{(a_2, a_1) \mid (a_1, a_2) \in P^{\mathcal{I}}\}$, We assume that Δ contains the constants and that $c^{\mathcal{I}} = c$ (i.e., we adopt *standard names*). An interpretation $\mathcal{I}$ is a *model* of a membership assertion $A(a)$ (resp., $\neg A(a)$) if $a \in A^{\mathcal{I}}$ (resp., $a \notin A^{\mathcal{I}}$), of $P(a, b)$ if $(a, b) \in P^{\mathcal{I}}$, of a (concept or role) inclusion assertion $D_1 \sqsubseteq D_2$ if $D_1^{\mathcal{I}} \subseteq D_2^{\mathcal{I}}$, and of a functionality assertion $(\mathsf{funct}\ R)$ if R is functional in $\mathcal{I}$, i.e., for every a there is at most one b such that $(a, b) \in R^{\mathcal{I}}$. It is often convenient to also view interpretations as sets of atoms and say that $A(a) \in \mathcal{I}$ iff $a \in A^{\mathcal{I}}$ and $P(a, b) \in \mathcal{I}$ iff $(a, b) \in P^{\mathcal{I}}$. If, under this view, $\mathcal{I}' \subseteq \mathcal{I}$, we say that $\mathcal{I}'$ is a *submodel* of $\mathcal{I}$.

As usual, we use $\mathcal{I} \models F$ to denote that $\mathcal{I}$ is a model of an assertion F, and for a KB $\mathcal{K}$, $\mathcal{I} \models \mathcal{K}$ denotes that $\mathcal{I} \models F$ for each assertion F in $\mathcal{K}$. We use *Mod*$(\mathcal{K})$ to denote the set of all models of $\mathcal{K}$. A KB is *satisfiable* if it has at least one model. A concept or role name X occurring in $\mathcal{K}$ is *coherent* if there is a model $\mathcal{I}$ of $\mathcal{K}$ such that $X^{\mathcal{I}} \neq \emptyset$. A KB $\mathcal{K}$ is *coherent* if every element of $\Sigma(\mathcal{K})$ is coherent. For *DL-Lite* the notions of satisfiability and coherence are tightly related: if $\mathcal{K}$ is coherent, then it is satisfiable. Indeed, consider a set of models $\mathcal{S}_{\mathcal{K}}$, containing one interpretation $\mathcal{I}_X$ for each $X \in \Sigma(\mathcal{K})$ such that $X^{\mathcal{I}_X} \neq \emptyset$ and $Y^{\mathcal{I}_X} = \emptyset$ for every $Y \neq X$, moreover, if $X \neq Y$, then $X^{\mathcal{I}_X}$ and in $Y^{\mathcal{I}_Y}$ are disjoint. One can easily see that the (disjoint) union of models in $\mathcal{S}_{\mathcal{K}}$ gives a models of $\mathcal{K}$. Further in the paper we consider *only* coherent (and consequently satisfiable) KBs.

This assumption does not affect the complexity results of this paper due to nice computational properties of the *DL-Lite* family, in particular w.r.t. the size of the extensional information (i.e., the data in the ABox). For example, KB coherence and satisfiability has polynomial-time complexity in the size of the TBox and logarithmic-space complexity[1] in the size of the ABox [2, 19].Note that tractability of satisfiability only holds for *DL-Lite*$_{\mathcal{FR}}$ under a syntactic restriction on interaction between functionality and role inclusion assertions: in a given TBox $\mathcal{T}$ every role R can either participate in a functionality assertion or in a role inclusion assertion but not in both, i.e., $(\mathsf{funct}\ R) \in \mathcal{T}$ if and only if both $R \sqsubseteq R' \notin \mathcal{T}$ and $R' \sqsubseteq R \notin \mathcal{T}$ holds for every R' different from R. In the following we assume that all *DL-Lite*$_{\mathcal{FR}}$ TBoxes satisfy this syntactic restriction.

[1] Actually, the data complexity of satisfiability and of other inference tasks that involve the ABox is AC^0

We use *entailment on KBs*, denoted $\mathcal{K} \models \mathcal{K}'$ in the standard sense, i.e., every model of $\mathcal{K}$ is also a model of $\mathcal{K}'$ (similarly for TBoxes and Ab) An ABox $\mathcal{A}$ $\mathcal{T}$*-entails* an ABox $\mathcal{A}'$, denoted $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}'$, if $\mathcal{T} \cup \mathcal{A} \models \mathcal{A}'$, and $\mathcal{A}$ is $\mathcal{T}$*-equivalent* to $\mathcal{A}'$, denoted $\mathcal{A} \equiv_{\mathcal{T}} \mathcal{A}'$, if $\mathcal{A} \models_{\mathcal{T}} \mathcal{A}'$ and $\mathcal{A}' \models_{\mathcal{T}} \mathcal{A}$. We say that $\mathcal{A}$ and $\mathcal{A}'$ are $\mathcal{T}$*-satisfiable* if $\mathcal{A} \cup \mathcal{A}' \not\models_{\mathcal{T}} \bot$, i.e., it does not imply falsehood (denoted using $\bot$).

The deductive *closure of a TBox* $\mathcal{T}$, denoted $cl(\mathcal{T})$, is the set of all TBox assertions F such that $\mathcal{T} \models F$. For a satisfiable KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, the *full closure of* $\mathcal{A}$ (wrt $\mathcal{T}$), denoted $fcl_{\mathcal{T}}(\mathcal{A})$, is the set of all membership assertions f (both positive and negative) over $adom(\mathcal{K})$ such that $\mathcal{A} \models_{\mathcal{T}} f$. In *DL-Lite*$_{\mathcal{FR}}$ both $cl(\mathcal{T})$ and $fcl_{\mathcal{T}}(\mathcal{A})$ are computable in time quadratic in, respectively, the number of assertions of $\mathcal{T}$, and $\mathcal{T} \cup \mathcal{A}$. Whenever needed we will assume w.l.o.g. that all TBoxes and ABoxes are closed.

A *homomorphism* μ from an interpretation $\mathcal{I}$ to an interpretation $\mathcal{J}$ over the same signature, is a structure-preserving mapping from Δ to Δ satisfying: *(i)* $\mu(a) = a$ for every constant a; *(ii)* if $x \in A^{\mathcal{I}}$ (resp., $(x, y) \in P^{\mathcal{I}}$), then $\mu(x) \in A^{\mathcal{J}}$ (resp., $(\mu(x), \mu(y)) \in P^{\mathcal{J}}$) for every atomic concept A (resp., atomic role P). We write $\mathcal{I} \hookrightarrow \mathcal{J}$ if there is a homomorphism from $\mathcal{I}$ to $\mathcal{J}$. A *canonical model* of $\mathcal{K}$, denoted $\mathcal{I}^{can}_{\mathcal{K}}$ or just $\mathcal{I}^{can}$ when $\mathcal{K}$ is clear from the context, is a model of $\mathcal{K}$ that can be homomorphically embedded in every model of $\mathcal{K}$.

Let $\mathcal{I}$ be an interpretation and $\mathcal{T}$ a *DL-Lite*$_{\mathcal{FR}}$ TBox. Then a *chase* of $\mathcal{I}$ wrt $\mathcal{T}$, denoted $chase_{\mathcal{T}}(\mathcal{I})$ (or just $chase(\mathcal{I})$ when $\mathcal{T}$ is clear), is an interpretation such that: *(i)* $\mathcal{I} \subseteq chase_{\mathcal{T}}(\mathcal{I})$, *(ii)* $chase_{\mathcal{T}}(\mathcal{I}) \models \mathcal{T}$, and *(iii)* for every interpretation $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{T}$ and $\mathcal{I} \subseteq \mathcal{J}$, it holds that $chase_{\mathcal{T}}(\mathcal{I}) \hookrightarrow \mathcal{J}$. The notion of chase can be defined also procedurally: go through all atoms $g \in \mathcal{I}$ and apply TBox inclusions to them, i.e., if $A(x) \in \mathcal{I}$, $(A \sqsubseteq B) \in \mathcal{T}$, and $B(x) \notin \mathcal{I}$, then $A \sqsubseteq B$ is applicable to $\mathcal{I}$ and the result of the application is the interpretation $\mathcal{I} \cup \{B(x)\}$. If $A(x) \in \mathcal{I}$, $(A \sqsubseteq \exists R) \in \mathcal{T}$, and there is no $y \in \Delta$ such that $R(x, y) \in \mathcal{I}$, then $A \sqsubseteq \exists R$ is applicable to $\mathcal{I}$ and the result of the application is $\mathcal{I} \cup \{(R(x, x_{new}))\}$, where x_{new} is fresh for $\mathcal{I}$, i.e., it does not occur before the application of the inclusion. The procedure terminates when no assertions of $\mathcal{T}$ are applicable to $\mathcal{I}$. One can find more details about the chase in [5]. Note that the way we define the chase is slightly different from the standard one for DL KBs (e.g., in [5]), in that the standard chase is built on ABoxes (their positive parts), which are essentially finite interpretations, while we define the chase on arbitrary (possibly infinite) models.

2.2. Evolution of Knowledge Bases

We introduce now formally the problem of ABox evolution of DL knowledge bases, concentrating on model-based approaches. We discuss different semantics for the problem and put them into relationship with each other. Specifically, we concentrate on the eight semantics that have been presented first in [8], and that are the result of considering the problem space according to three orthogonal dimensions (see Figure 1, right). Notice that the notions we introduce do not depend on any specific DL, although we will apply them later to the case of *DL-Lite*$_{\mathcal{FR}}$.

Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL KB and $\mathcal{N}$ a "new" ABox. We study how to incorporate assertions of $\mathcal{N}$ into $\mathcal{K}$, that is, how $\mathcal{K}$ *evolves* under $\mathcal{N}$ [12]. More practically, we study *evolution operators* that take $\mathcal{K}$ and $\mathcal{N}$ as input and return, possibly in *polynomial time*, a DL KB $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$ (with the same TBox as that of $\mathcal{K}$) that captures the evolution, and that we call *the (ABox) evolution of* $\mathcal{K}$ *under* $\mathcal{N}$. Based on the evolution principles of [8], we require $\mathcal{K}$ and $\mathcal{K}'$ to be satisfiable. A DL KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and an ABox $\mathcal{N}$ is called an *evolution setting* if both $\mathcal{K}$ and $\mathcal{T} \cup \mathcal{N}$ are satisfiable and $\mathcal{N}$ contains ABox assertions of the form $A(a)$ and $R(a, b)$ only. In this paper we concentrate on *DL-Lite*$_{\mathcal{FR}}$ evolution settings, i.e., those in which both $\mathcal{K}$ and $\mathcal{N}$ are expressed in *DL-Lite*$_{\mathcal{FR}}$.

Model-Based Semantics of Evolution. In model-based approaches (MBAs), the result of evolution of a KB $\mathcal{K}$ wrt new knowledge $\mathcal{N}$ is a set $\mathcal{K} \diamond \mathcal{N}$ of models. The idea of MBAs is to choose as $\mathcal{K} \diamond \mathcal{N}$ some models

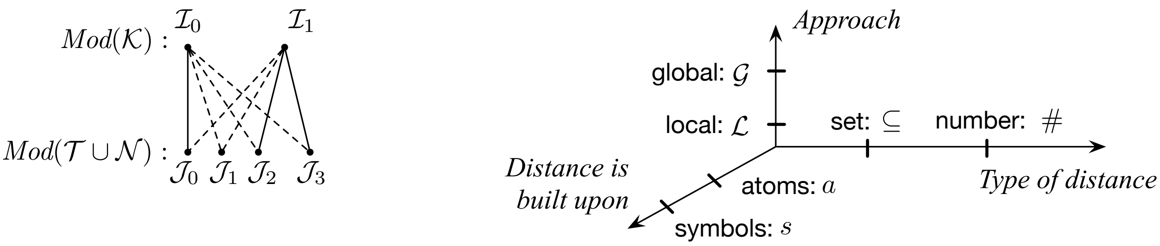

Figure 1: Left: measuring distances between models and finding local minimums.
Right: three-dimensional space of approaches to model-based evolution semantics.

of $\mathcal{T} \cup \mathcal{N}$ depending on their distance to the models of $\mathcal{K}$. Katsuno and Mendelzon [13] considered two ways, so called *local* and *global*, of choosing these models of $\mathcal{T} \cup \mathcal{N}$, where the former choice corresponds to *knowledge update* and the latter one to *knowledge revision*.

The idea of the local approaches is to consider all models of $\mathcal{K}$ one by one, and for each model $\mathcal{I}$ to take those models $\mathcal{J}$ of $\mathcal{T} \cup \mathcal{N}$ that are minimally distant from $\mathcal{I}$. Formally,

$$\mathcal{K} \diamond \mathcal{N} = \bigcup_{\mathcal{I} \in Mod(\mathcal{K})} \mathcal{I} \diamond \mathcal{N}, \text{ where } \mathcal{I} \diamond \mathcal{N} = \operatorname*{arg\,min}_{\mathcal{J} \in Mod(\mathcal{T} \cup \mathcal{N})} dist(\mathcal{I}, \mathcal{J}),$$

where $dist(\cdot, \cdot)$ is a function measuring the distance between two models, and $\arg\min$ stands for the *argument of the minimum,* that is, in our case, the set of models $\mathcal{J}$ for which the value of $dist(\mathcal{I}, \mathcal{J})$ reaches its minimum value, given $\mathcal{I}$. The distance function *dist* varies from approach to approach and commonly takes as values either numbers or subsets of some fixed set.

Example 1. To get a better intuition of the local semantics, consider Figure 1, left, where we present two models $\mathcal{I}_0$ and $\mathcal{I}_1$ of a KB $\mathcal{K}$ and four models $\mathcal{J}_0, \ldots, \mathcal{J}_3$ of $\mathcal{T} \cup \mathcal{N}$. We represent the distance between a model of $\mathcal{K}$ and a model of $\mathcal{T} \cup \mathcal{N}$ by the length of the line connecting them. Solid lines correspond to minimal distances, dashed lines to distances that are not minimal. In this figure $\{\mathcal{J}_0\} = \arg\min_{\mathcal{J} \in \{\mathcal{J}_0, \ldots, \mathcal{J}_3\}} dist(\mathcal{I}_0, \mathcal{J})$ and $\{\mathcal{J}_2, \mathcal{J}_3\} = \arg\min_{\mathcal{J} \in \{\mathcal{J}_0, \ldots, \mathcal{J}_3\}} dist(\mathcal{I}_1, \mathcal{J})$. ■

In the global approach one chooses models of $\mathcal{T} \cup \mathcal{N}$ that are minimally distant from $\mathcal{K}$:

$$\mathcal{K} \diamond \mathcal{N} = \operatorname*{arg\,min}_{\mathcal{J} \in Mod(\mathcal{T} \cup \mathcal{N})} dist(Mod(\mathcal{K}), \mathcal{J}), \tag{1}$$

where $dist(Mod(\mathcal{K}), \mathcal{J}) = \min_{\mathcal{I} \in Mod(\mathcal{K})} dist(\mathcal{I}, \mathcal{J})$.

Example 2. Consider again Figure 1, left, and assume that the distance between $\mathcal{I}_0$ and $\mathcal{J}_0$ is the global minimum. Thus, we obtain that $\{\mathcal{J}_0\} = \arg\min_{\mathcal{J} \in \{\mathcal{J}_0, \ldots, \mathcal{J}_3\}} dist(\{\mathcal{I}_0, \mathcal{I}_1\}, \mathcal{J})$. ■

Measuring Distance Between Interpretations. The classical MBAs were developed for propositional theories [11], where interpretations are sets of propositional atoms. In that case, two distance functions have been introduced, respectively based on symmetric difference "$\ominus$" and on the cardinality of symmetric difference:

$$dist_{\subseteq}(\mathcal{I}, \mathcal{J}) = \mathcal{I} \ominus \mathcal{J} \qquad \text{and} \qquad dist_{\#}(\mathcal{I}, \mathcal{J}) = |\mathcal{I} \ominus \mathcal{J}|, \tag{2}$$

where $\mathcal{I} \ominus \mathcal{J} = (\mathcal{I} \setminus \mathcal{J}) \cup (\mathcal{J} \setminus \mathcal{I})$. Distances under $dist_{\subseteq}$ are sets and are compared by set inclusion, that is, $dist_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \leq dist_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2)$ if and only if $dist_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \subseteq dist_{\subseteq}(\mathcal{I}_2, \mathcal{J}_2)$. Finite distances under $dist_{\#}$ are natural numbers and are compared in the standard way.

One can extend these distances to DL interpretations in two different ways. One way is to consider interpretations $\mathcal{I}$, $\mathcal{J}$ as sets of *atoms*. Then $\mathcal{I} \ominus \mathcal{J}$ is again a set of atoms and we can define distances as in Equation (2). We denote these distances as $dist^a_{\subseteq}(\mathcal{I}, \mathcal{J})$ and $dist^a_{\#}(\mathcal{I}, \mathcal{J})$. While in the propositional case distances are always finite, note that this is not the case for DL interpretations, which may be infinite. Another way is to define distances at the level of the concept and role *symbols* in the signature Σ underlying the interpretations:

$$dist^s_{\subseteq}(\mathcal{I}, \mathcal{J}) = \{S \in \Sigma \mid S^{\mathcal{I}} \neq S^{\mathcal{J}}\} \qquad \text{and} \qquad dist^s_{\#}(\mathcal{I}, \mathcal{J}) = |\{S \in \Sigma \mid S^{\mathcal{I}} \neq S^{\mathcal{J}}\}|.$$

Summing up across the different possibilities, we have three dimensions with two values each: (1) the *local* or the *global* approach, (2) *atoms* or *symbols* for defining distances, and (3) *set inclusion* or *cardinality* to compare symmetric differences. This gives eight semantics of evolution according to MBAs, as shown in Figure 1, right. By $\mathcal{L}$ we denote local and by $\mathcal{G}$ global semantics. We attach the superscripts a or s to indicate whether distances are defined in terms of atoms or symbols. We use the subscripts $\subseteq$ or $\#$ to indicate whether distances are compared in terms of set inclusion or cardinality. For example, $\mathcal{L}^a_{\#}$ denotes the local semantics where the distances are in terms of cardinality of sets of atoms.

Under each of the eight semantics we consider, evolution results in a set of interpretations. In the propositional case, each set of interpretations over finitely many symbols can be captured by a suitable formula, that is, a formula whose models are exactly those interpretations. In the case of DLs this is no more necessarily the case, since on the one hand, interpretations can be infinite, and on the other hand logics may miss some connectives like disjunction or negation that would be necessary to capture those interpretations.

Closure Under Evolution. Let $\mathcal{D}$ be a DL and $\mathcal{M}$ a set of models. We say that $\mathcal{M}$ is *axiomatizable* in $\mathcal{D}$ if there is a KB $\mathcal{K}$ such that $\mathit{Mod}(\mathcal{K}) = \mathcal{M}$.

Definition 3 (Closure under Evolution)**.** Let S be one of the eight MBAs presented above. We say that $\mathcal{D}$ is *closed under evolution wrt a semantics* S (or, evolution wrt S is *expressible* in $\mathcal{D}$) if for every evolution setting $\mathcal{K}$ and $\mathcal{N}$ expressed in $\mathcal{D}$, there is a KB $\mathcal{K}'$ expressed in $\mathcal{D}$ such that $\mathit{Mod}(\mathcal{K}') = \mathcal{K} \diamond \mathcal{N}$, where $\mathcal{K} \diamond \mathcal{N}$ is the evolution result under S. ■

It has been shown in [8, 9] that *DL-Lite* is not closed under any of the eight model based semantics. The observation underlying these results is that on the one hand, the minimality of change principle intrinsically introduces implicit disjunction in the evolved KB. On the other hand, since *DL-Lite* is a slight extension of Horn logic [7], it does not allow one to express genuine disjunction (see Lemma 1 in [8] for details).

Let $\mathcal{M}$ be a set of models that resulted from the evolution of $\mathcal{T} \cup \mathcal{A}$ with $\mathcal{N}$. A KB $\mathcal{T} \cup \mathcal{A}'$ is a *sound approximation* of $\mathcal{M}$ if $\mathcal{M} \subseteq \mathit{Mod}(\mathcal{T} \cup \mathcal{A}')$. A sound approximation $\mathcal{T} \cup \mathcal{A}'$ is *minimal* if there is no sound approximation $\mathcal{T} \cup \mathcal{A}''$ of $\mathcal{M}$ such that $\mathit{Mod}(\mathcal{T} \cup \mathcal{A}'') \subsetneq \mathit{Mod}(\mathcal{T} \cup \mathcal{A}')$, i.e., $\mathcal{T} \cup \mathcal{A}'$ is minimal wrt "$\subseteq$".

2.3. Running Example

In the following example we present a KB that we will use to illustrate our results.

Example 4. Consider an ABox $\mathcal{A}^{ex}$:

$$\mathcal{A}^{ex} = \{\mathit{Priest}(\mathit{pedro}),\ \mathit{Priest}(\mathit{ivan}),\ \mathit{Husb}(\mathit{john}),\ \mathit{Wife}(\mathit{mary}),\ \mathit{Wife}(\mathit{chloe}),\ \mathit{HasHusb}(\mathit{mary}, \mathit{john})\}.$$

Intuitively, $\mathcal{A}^{ex}$ says that there are two priests pedro and ivan, one husband john, two wives mary and chloe, and mary is in the has-husband relationship with john. Consider a TBox $\mathcal{T}^{ex}$:

$$\mathcal{T}^{ex} = \{\mathit{Card} \sqsubseteq \mathit{Priest},\ \exists \mathit{HasHusb}^- \sqsubseteq \neg \mathit{Priest},\ \mathit{Husb} \sqsubseteq \exists \mathit{HasHusb}^-,\ \mathit{Wife} \sqsubseteq \exists \mathit{HasHusb}\}.$$

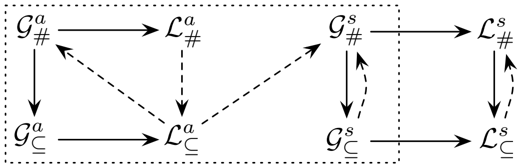

Figure 2: Subsumptions for evolution semantics. The arrows stand for the subsumption $\preccurlyeq_{sem}$: " $\longrightarrow$ ": for *DL-Lite*$_{\mathcal{R}}$ (Theorem 7). " $\dashrightarrow$ ": for *DL-Lite*$_{\mathcal{R}}^{pr}$ (Theorems 18, 20). The dashed frame surrounds those semantics under which *DL-Lite*$_{\mathcal{R}}^{pr}$ is closed.

Intuitively, $\mathcal{T}^{ex}$ says that cardinals are priests, husbands and fillers of the $HasHusb$ role are not priests, wives and husbands are those participating in the $HasHusb$ relationship. In the remainder of the paper we will show how different examples of new knowledge affect the original ABox. Examples of new knowledge are:

$$\mathcal{N}_1^{ex} = \{Priest(john)\} \qquad \text{and} \qquad \mathcal{N}_2^{ex} = \{Husb(pedro),\ Wife(tanya)\}. \qquad \blacksquare$$

3. Relationships Between Model-Based Semantics

In this section, we define a framework for comparing different model-based evolution semantics and apply it to the eight semantics that have been presented in Section 2.2.

Definition 5 (Subsumption on Evolution Semantics)**.** Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two evolution semantics, $\mathcal{D}$ a logic language, and let $\mathcal{K} \diamond_{\mathcal{S}} \mathcal{N}$ denote evolution under a semantics $\mathcal{S}$. Then $\mathcal{S}_1$ *is subsumed by* $\mathcal{S}_2$ wrt $\mathcal{D}$, denoted $(\mathcal{S}_1 \preccurlyeq_{sem} \mathcal{S}_2)(\mathcal{D})$, or just $\mathcal{S}_1 \preccurlyeq_{sem} \mathcal{S}_2$ when $\mathcal{D}$ is clear from the context, if $\mathcal{K} \diamond_{\mathcal{S}_1} \mathcal{N} \subseteq \mathcal{K} \diamond_{\mathcal{S}_2} \mathcal{N}$ for all evolution settings $\mathcal{K}$ and $\mathcal{N}$ in $\mathcal{D}$. $\mathcal{S}_1$ and $\mathcal{S}_2$ are *equivalent* wrt $\mathcal{D}$, denoted $(\mathcal{S}_1 \equiv_{sem} \mathcal{S}_2)(\mathcal{D})$, if $(\mathcal{S}_1 \preccurlyeq_{sem} \mathcal{S}_2)(\mathcal{D})$ and $(\mathcal{S}_2 \preccurlyeq_{sem} \mathcal{S}_1)(\mathcal{D})$. ■

The following theorem shows the subsumption relation between different semantics, independently of the chosen DL. We depict these relations in Figure 2 using solid arrows.

Theorem 6. *Let $\beta \in \{a, s\}$ and $\alpha \in \{\subseteq, \#\}$. Then for any DL it holds that*

$$\mathcal{G}_\alpha^\beta \preccurlyeq_{sem} \mathcal{L}_\alpha^\beta, \qquad \mathcal{L}_\#^s \preccurlyeq_{sem} \mathcal{L}_\subseteq^s, \qquad \textit{and} \qquad \mathcal{G}_\#^s \preccurlyeq_{sem} \mathcal{G}_\subseteq^s. \tag{3}$$

Proof.
$\mathcal{G}_\alpha^\beta \preccurlyeq_{sem} \mathcal{L}_\alpha^\beta$: Let $dist_\alpha^\beta$ be a distance function, and let $\mathcal{E}_\mathcal{G} = \mathcal{K} \diamond \mathcal{N}$ wrt $\mathcal{G}_\alpha^\beta$ and $\mathcal{E}_\mathcal{L} = \mathcal{K} \diamond \mathcal{N}$ wrt $\mathcal{L}_\alpha^\beta$ respectively be the global and local semantics based on $dist_\alpha^\beta$. For an evolution setting $\mathcal{K}$ and $\mathcal{N}$, let $\mathcal{J}' \in \mathcal{E}_\mathcal{G}$. Then, there is $\mathcal{I}' \models \mathcal{K}$ such that for every $\mathcal{I}'' \models \mathcal{K}$ and $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ it *does not* hold that

$$dist_\alpha^\beta(\mathcal{I}'', \mathcal{J}'') \lneq dist_\alpha^\beta(\mathcal{I}', \mathcal{J}').$$

In particular, when $\mathcal{I}'' = \mathcal{I}'$, there is no $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ such that $dist_\alpha^\beta(\mathcal{I}', \mathcal{J}'') \lneq dist_\alpha^\beta(\mathcal{I}', \mathcal{J}')$, which yields that $\mathcal{J}' \in \arg\min_{\mathcal{J} \in Mod(\mathcal{T} \cup \mathcal{N})} dist_\alpha^\beta(\mathcal{I}', \mathcal{J})$, and hence $\mathcal{J}' \in \mathcal{E}_\mathcal{L}$, which concludes the proof.

$\mathcal{L}_\#^s \preccurlyeq_{sem} \mathcal{L}_\subseteq^s$: Consider $\mathcal{E}_\# = \mathcal{K} \diamond \mathcal{N}$ wrt $\mathcal{L}_\#^s$, which is based on the distance $dist_\#^s$, and $\mathcal{E}_\subseteq = \mathcal{K} \diamond \mathcal{N}$ wrt $\mathcal{L}_\subseteq^s$, which is based on $dist_\subseteq^s$. We now are interested in establishing whether $\mathcal{E}_\# \subseteq \mathcal{E}_\subseteq$ holds. Assume $\mathcal{J}' \in \mathcal{E}_\#$ and $\mathcal{J}' \notin \mathcal{E}_\subseteq$. Then, from the former assumption we conclude existence of $\mathcal{I}' \models \mathcal{K}$ such that $\mathcal{J}' \in \arg\min_{\mathcal{J} \in Mod(\mathcal{T} \cup \mathcal{N})} dist_\#^s(\mathcal{I}', \mathcal{J})$. From the latter assumption, $\mathcal{J}' \notin \mathcal{E}_\subseteq$, we conclude existence of a model $\mathcal{J}''$ such that $dist_\subseteq^s(\mathcal{I}', \mathcal{J}'') \subsetneq dist_\subseteq^s(\mathcal{I}', \mathcal{J}')$. Since the signature of $\mathcal{K} \cup \mathcal{N}$ is finite, the distance $dist_\subseteq^s$ between every two models over this signature is also finite. Thus, we obtain that $dist_\#(\mathcal{I}', \mathcal{J}'') \lneq dist_\#(\mathcal{I}', \mathcal{J}')$, which contradicts to $\mathcal{J}' \in \mathcal{E}_\#$ and concludes the proof. The case $\mathcal{G}_\#^s \preccurlyeq_{sem} \mathcal{G}_\subseteq^s$ is analogous. □

For *DL-Lite*$_{\mathcal{FR}}$, Figure 2 is complete in the following sense: there is a solid oriented path (a sequence of solid arrows) from a semantics $\mathcal{S}_1$ to a semantics $\mathcal{S}_2$ if and only if $\mathcal{S}_1 \preccurlyeq_{sem} \mathcal{S}_2$. This is formally stated in the following theorem, whose proof we omit for lack of space.

Theorem 7. *For DL-Lite*$_{\mathcal{FR}}$ *the relation* $\mathcal{G}^a_{\#} \preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$ *holds. Moreover, let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be two different model-based semantics from* $\{\mathcal{X}^\beta_\alpha \mid \mathcal{X} \in \{\mathcal{L}, \mathcal{G}\}, \beta \in \{a, s\}, \alpha \in \{\subseteq, \#\}\}$*. Then, for DL-Lite*$_{\mathcal{FR}}$*,* $\mathcal{S}_1 \not\preccurlyeq_{sem} \mathcal{S}_2$ *holds unless* $\mathcal{S}_1 \preccurlyeq_{sem} \mathcal{S}_2$ *is in the reflexive and transitive closure of the relations from Equation 3 and* $\mathcal{G}^s_{\#} \preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$*.*

Note that some of the negative cases for symbol-based MBAs already hold on a restriction of *DL-Lite*$_{\mathcal{FR}}$, called *DL-Lite*$^{pr}_{\mathcal{R}}$, which we will introduce in the following section. Thus the proof of Theorem 20 gives proofs for the negative cases $\mathcal{G}^s_{\#} \not\preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$ and $\mathcal{L}^s_{\subseteq} \not\preccurlyeq_{sem} \mathcal{G}^s_{\#}$ of the current theorem.

4. Evolution of *DL-Lite*$^{pr}_{\mathcal{R}}$ KBs

In this section, we consider a restriction of *DL-Lite*$_{\mathcal{R}}$, which we call *DL-Lite*$^{pr}_{\mathcal{R}}$ (pr stands for *positive role* interaction). Intuitively, in *DL-Lite*$^{pr}_{\mathcal{R}}$ disjointness that involves roles is forbidden, so that only positive inclusions involving roles are permitted.

Definition 8 (The DL *DL-Lite*$^{pr}_{\mathcal{R}}$)**.** A *DL-Lite*$_{\mathcal{R}}$ knowledge base $\mathcal{K}$ is in *DL-Lite*$^{pr}_{\mathcal{R}}$ if there is *no* role R, basic concept B and constants a, b such that

$$\mathcal{K} \models \neg\exists R(a) \quad \text{and} \quad \mathcal{K} \models \exists R \sqsubseteq \neg B. \qquad \blacksquare$$

DL-Lite$^{pr}_{\mathcal{R}}$ is defined semantically, but one can syntactically check (in quadratic time), given a *DL-Lite*$_{\mathcal{R}}$ TBox $\mathcal{T}$, whether it is in *DL-Lite*$^{pr}_{\mathcal{R}}$: compute the closure of $\mathcal{T}$, and check that no assertion of the form $\exists R \sqsubseteq \neg B$ is in the closure. If this is the case, then $\mathcal{K}$ is in *DL-Lite*$^{pr}_{\mathcal{R}}$.

Example 9. Consider again $\mathcal{T}^{ex}$ of Example 4. To see that $\mathcal{T}^{ex}$ is not in *DL-Lite*$^{pr}_{\mathcal{R}}$ observe that $\mathcal{T}^{ex} \models \exists HasHusb^- \sqsubseteq \neg Priest$. Thus, consider the following subset of $\mathcal{T}^{ex}$, which does not entail the assertion $\exists HasHusb^- \sqsubseteq \neg Priest$:

$$\mathcal{T}^{ex}_1 = \{Card \sqsubseteq Priest,\ Husb \sqsubseteq \neg Priest\}.$$

Note that $\mathcal{T}^{ex}_1$ misses more assertions of $\mathcal{T}^{ex}$ than just the one mentioned. This is done only for the sake of simplification of the upcoming examples. ■

We see *DL-Lite*$^{pr}_{\mathcal{R}}$ as an important language to study because it is an extension of the RDFS ontology language [21] (more precisely, of the first-order logic fragment of RDFS). *DL-Lite*$^{pr}_{\mathcal{R}}$ adds to RDFS the ability of expressing disjointness of atomic concepts ($A_1 \sqsubseteq \neg A_2$) and mandatory participation to roles ($A \sqsubseteq \exists R$). Also note that since *DL-Lite*$^{pr}_{\mathcal{R}}$ restricts *DL-Lite*$_{\mathcal{R}}$ (and consequently *DL-Lite*$_{\mathcal{FR}}$), the relations between MBAs from Theorems 6 and 7 are correct also for *DL-Lite*$^{pr}_{\mathcal{R}}$. In the rest of the section we consider *DL-Lite*$^{pr}_{\mathcal{R}}$ KBs $\mathcal{K}$ and investigate whether it is possible and how to capture $\mathcal{K} \diamond \mathcal{N}$ in *DL-Lite*$^{pr}_{\mathcal{R}}$ under all eight MBAs. We start with the atom-based approaches.

4.1. Capturing Atom-Based Evolution

Consider the algorithm `AlignAlg` presented in Algorithm 1, which takes as input an evolution setting $\mathcal{K}$ and $\mathcal{N}$, and returns the maximal subset of $fcl_{\mathcal{T}}(\mathcal{A})$ that is $\mathcal{T}$-satisfiable with $\mathcal{N}$.

Algorithm 1: AlignAlg($\mathcal{T} \cup \mathcal{A}, \mathcal{N}$) computing an ABox $\mathcal{A}'$ to represent the result of evolution

INPUT : evolution setting $\mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ with $\mathcal{T}$ in *DL-Lite*$_{\mathcal{R}}^{pr}$

OUTPUT: he maximal set $\mathcal{A}' \subseteq \mathit{fcl}_{\mathcal{T}}(\mathcal{A})$ of ABox assertions that is $\mathcal{T}$-satisfiable with $\mathcal{N}$

$\mathcal{A}' := \emptyset$; $\quad \mathcal{S} := \mathit{fcl}_{\mathcal{T}}(\mathcal{A})$;
repeat
 choose some $\phi \in \mathcal{S}$; $\quad \mathcal{S} := \mathcal{S} \setminus \{\phi\}$;
 if $\{\phi\} \cup \mathcal{N} \not\models_{\mathcal{T}} \bot$ **then** $\mathcal{A}' := \mathcal{A}' \cup \{\phi\}$;
until $\mathcal{S} = \emptyset$;
return $\mathcal{A}'$;

Example 10. We illustrate how AlignAlg works on $\mathcal{T}_1^{ex} \cup \mathcal{A}^{ex}$ and both $\mathcal{N}_1^{ex}$ and $\mathcal{N}_2^{ex}$. The full $\mathcal{T}$-closure of $\mathcal{A}^{ex}$ with $\mathcal{T}_1^{ex}$ is $\mathit{fcl}_{\mathcal{T}_1^{ex}}(\mathcal{A}^{ex}) = \mathcal{A}^{ex} \cup \{\neg \mathit{Husb}(\mathit{pedro}),\ \neg \mathit{Husb}(\mathit{ivan}),\ \neg \mathit{Priest}(\mathit{john}),\ \neg \mathit{Card}(\mathit{john})\}$. Thus,

$$\texttt{AlignAlg}(\mathcal{T}_1^{ex} \cup \mathcal{A}^{ex}, \mathcal{N}_1^{ex}) = \mathit{fcl}_{\mathcal{T}_1^{ex}}(\mathcal{A}^{ex}) \setminus \{\mathit{Husb}(\mathit{john}),\ \neg \mathit{Priest}(\mathit{john})\}.$$

For $\mathcal{N}_2^{ex}$ we have AlignAlg($\mathcal{T}_1^{ex} \cup \mathcal{A}^{ex}, \mathcal{N}_2^{ex}$) $= \mathit{fcl}_{\mathcal{T}_1^{ex}}(\mathcal{A}^{ex}) \setminus \{\mathit{Priest}(\mathit{pedro}),\ \neg \mathit{Husb}(\mathit{pedro})\}$. ■

We are going to prove that in *DL-Lite*$_{\mathcal{R}}^{pr}$ using AlignAlg one can efficiently compute a representation of the evolution result $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}_{\subseteq}^{a}$. We first illustrate this with an example.

Example 11. Continuing with Example 10 and due to Theorem 12 we have that for $\mathcal{N}_1^{ex}$:

$$\mathcal{K}' = \mathcal{T}_1^{ex} \cup \mathit{fcl}_{\mathcal{T}_1^{ex}}(\mathcal{A}^{ex}) \setminus \{\mathit{Husb}(\mathit{john}),\ \neg \mathit{Priest}(\mathit{john})\} \cup \{\mathit{Priest}(\mathit{john})\}.$$

This result is quite intuitive and expected: the new knowledge $\mathcal{N}_1^{ex}$ asserts that john is a priest now, while the TBox $\mathcal{T}_1^{ex}$ forbids to be a priest and a husband at once, thus, we have to drop from the old knowledge that john is a husband and that he is not a priest. Also note that $\mathcal{K}'$ contains $\neg \mathit{Card}(\mathit{john})$, that is, the fact that john became a priest did not make him a cardinal. As for $\mathcal{N}_2^{ex}$, the result is:

$$\mathcal{K}' = \mathcal{T}_1^{ex} \cup \Big(\mathit{fcl}_{\mathcal{T}_1^{ex}}(\mathcal{A}^{ex}) \setminus \{\mathit{Priest}(\mathit{pedro}),\ \neg \mathit{Husb}(\mathit{pedro})\} \cup \{\mathit{Husb}(\mathit{pedro}),\ \mathit{Wife}(\mathit{tanya})\}\Big).$$

This result is again expected: pedro becomes a husband and we have to drop the old knowledge that he is a priest and not a husband. Moreover, tanya becomes a wife and, since this fact does not conflict with anything in the old knowledge, we just add it. ■

We are now ready to state our result formally.

Theorem 12. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ be an evolution setting, and $\mathcal{T}$ in DL-Lite$_{\mathcal{R}}^{pr}$. Then*

$$\mathcal{K}' = \mathcal{T} \cup \mathit{AlignAlg}(\mathcal{K}, \mathcal{N}) \cup \mathcal{N}. \tag{4}$$

is a DL-Lite$_{\mathcal{R}}^{pr}$ representation of $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}_{\subseteq}^{a}$. Moreover, $\mathcal{K}'$ is computable in time polynomial in $|\mathcal{K} \cup \mathcal{N}|$.

As a corollary of the theorem note that for *DL-Lite*$_{\mathcal{R}}^{pr}$, Bold Semantics introduced in [8] (which defines $\mathcal{T} \cup \mathcal{A} \diamond \mathcal{N}$ as $\mathcal{T} \cup \mathcal{A}''$, where $\mathcal{A}''$ is a maximal subset of $\mathit{cl}_{\mathcal{T}}(\mathcal{A})$ satisfiable with $\mathcal{N}$) can be seen as a syntactical counterpart of $\mathcal{L}_{\subseteq}^{a}$, if one extends Bold Semantics by considering $\mathit{fcl}_{\mathcal{T}}(\mathcal{A})$ instead of $\mathit{cl}_{\mathcal{T}}(\mathcal{A})$, that is, if one also accounts for negative assertions $\mathcal{T}$-entailed from $\mathcal{A}$.

Before proving the theorem, we present a lemma that will help us for a given model $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$ to construct a model $\mathcal{I} \models \mathcal{K}$ such that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$. This method of constructing $\mathcal{I}$ is the key for proving most of the theorems in this section. Intuitively, $\mathcal{I}$ is constructed in two steps: *(i)* by dropping from $\mathcal{J}$ all unary atomic MAs (i.e., unary atoms) that are *not* $\mathcal{T}$-satisfiable with $\mathcal{A}$, and then *(ii)* adding unary atomic MAs that are $\mathcal{T}$-entailed from $\mathcal{A}$. Let $\mathcal{T}$ be a TBox, $\mathcal{A}$ an ABox satisfiable with $\mathcal{T}$, then the *unary closure* of $\mathcal{A}$ wrt $\mathcal{T}$ is

$$\mathit{ucl}_\mathcal{T}(\mathcal{A}) = \{A(c) \mid A \text{ is an atomic concept, } c \text{ is a constant, and } \mathcal{A} \models_\mathcal{T} A(c)\}.$$

Let $\mathcal{J}$ be an interpretation, then

$$\mathcal{J}^\mathcal{A} = \{A(c) \in \mathcal{J} \mid A \text{ is an atomic concept, } c \text{ is a constant, and } \mathcal{A} \cup \{A(c)\} \models_\mathcal{T} \bot\}.$$

Observe that the sets $\mathit{ucl}_\mathcal{T}(\mathcal{A})$ and $\mathcal{J}^\mathcal{A}$ are finite. Indeed, if $\mathcal{A} \models_\mathcal{T} A(c)$ or $\mathcal{A} \cup \{A(c)\} \models_\mathcal{T} \bot$ then one can easily show that both A and c occur in $\mathcal{A}$. Due to finiteness of $\mathcal{A}$, the sets $\mathit{ucl}_\mathcal{T}(\mathcal{A})$ and $\mathcal{J}^\mathcal{A}$ are also finite. If $\mathcal{I}$ is an interpretation and g is a positive MA, then $\mathcal{I}[g]$ is a minimal submodel of $\mathcal{I}$ satisfying both g and $\mathcal{T}$.

Lemma 13. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ be an evolution setting, and $\mathcal{K}$ in DL-Lite$^{pr}_\mathcal{R}$. Let $\mathcal{J}$ be a model of $\mathcal{T} \cup \texttt{AlignAlg}(\mathcal{K},\mathcal{N}) \cup \mathcal{N}$. Then the following interpretation is a model of $\mathcal{K}$:*

$$\mathcal{I} = (\mathcal{J} \setminus \mathcal{J}^\mathcal{A}) \cup \mathit{ucl}_\mathcal{T}(\mathcal{A}). \tag{5}$$

Moreover, $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$ and $\mathit{dist}^a_\subseteq(\mathcal{I},\mathcal{J})$ is finite.

Proof. Finiteness of $\mathit{dist}^a_\subseteq(\mathcal{I},\mathcal{J})$ follows from finiteness of $\mathcal{J}^\mathcal{A}$ and $\mathit{ucl}_\mathcal{T}(\mathcal{A})$. We first prove that $\mathcal{I} \models \mathcal{K}$ by showing that $\mathcal{I} \models \mathcal{A}$ and $\mathcal{I} \models \mathcal{T}$. Afterwards we will prove that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{K}$ under $\mathcal{L}^a_\subseteq$. Let $\mathcal{A}' = \texttt{AlignAlg}(\mathcal{K},\mathcal{N}) \cup \mathcal{N}$.

$\mathcal{I} \models \mathcal{A}$: Since $\mathit{ucl}_\mathcal{T}(\mathcal{A}) \subseteq \mathcal{I}$, we have $\mathcal{I} \models A(c)$ for all MAs of the form $A(c) \in \mathcal{A}$. It remains to show that $\mathcal{I} \models g$, for all MAs g of the form $R(a,b)$ or $\exists R(a)$ from $\mathcal{A}$. This follows from the following observation and the way we construct $\mathcal{I}$:

Proposition 14. *For a DL-Lite$^{pr}_\mathcal{R}$ KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and an assertion g of the form $R(a,b)$ or $\exists R(a)$, if $\mathcal{A} \models_\mathcal{T} g$, then also $\texttt{AlignAlg}(\mathcal{K},\mathcal{N}) \cup \mathcal{N} \models_\mathcal{T} g$.*

Indeed, from $\mathcal{J} \models \mathcal{A}'$ we conclude $\mathcal{J} \models g$ for g's as above. Since $\mathcal{J}^\mathcal{A}$ includes only assertions of the form $A(c)$ for atomic concepts A, no such g is dropped from $\mathcal{J}$ while constructing $\mathcal{I}$, thus, $\mathcal{I} \models g$.

$\mathcal{I} \models \mathcal{T}$: We now show that $\mathcal{I} \models \mathcal{T}$ in two steps. First observe that $\mathcal{J} \setminus \mathcal{J}^\mathcal{A} \models \mathcal{T}$. Indeed, since $\mathcal{J} \models \mathcal{T}$, it is enough to show for every $f \in \mathcal{J}^\mathcal{A}$, if $f' \models_\mathcal{T} f$ for some $f' \in \mathcal{J}$, then $f' \in \mathcal{J}^\mathcal{A}$. This is clearly the case because $f' \models_\mathcal{T} f$ and $\mathcal{A} \cup \{f\} \models_\mathcal{T} \bot$ imply $\mathcal{A} \cup \{f'\} \models_\mathcal{T} \bot$ and, consequently, $f' \in \mathcal{J}^\mathcal{A}$. Now we show that adding $\mathit{ucl}_\mathcal{T}(\mathcal{A})$ to $\mathcal{J} \setminus \mathcal{J}^\mathcal{A}$ does not violate $\mathcal{T}$. Indeed, let f be an assertion from $\mathit{ucl}_\mathcal{T}(\mathcal{A})$, we need to show that for every assertion g such that $f \models_\mathcal{T} g$ it holds $\mathcal{I} \models g$. Clearly, g can only be of the form either $A(c)$, or $\exists R(a)$. If $g = A(c)$, then $g \in \mathit{ucl}_\mathcal{T}(\mathcal{A})$ and obviously $\mathcal{I} \models g$. If $g = \exists R(a)$, then, observe that $f \models_\mathcal{T} g$ implies $\mathcal{A} \models_\mathcal{T} g$, thus due to Proposition 14, $\mathcal{A}' \models g$ and, as we saw above, $\mathcal{I} \models g$.

$\mathcal{J} \in \mathcal{I} \diamond \mathcal{K}$ under $\mathcal{L}^a_\subseteq$: By the definition of $\mathcal{L}^a_\subseteq$ semantics, we should prove that there is *no* $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ such that $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$. Assume there exists such $\mathcal{J}'$. Thus, there is an atom f such that $f \notin \mathcal{I} \ominus \mathcal{J}'$ while $f \in \mathcal{I} \ominus \mathcal{J}$. By definition of $\mathcal{I}$, interpretations $\mathcal{I}$ and $\mathcal{J}$ differ only on atoms of the form $A(c)$, hence, f is of the form $A(c)$ (it cannot be of the form $R(a,b)$). We have two cases:

(i) $A(c) \in \mathcal{I} \setminus \mathcal{J}$ and $A(c) \in \mathcal{J}'$: By construction of $\mathcal{I}$, $A(c) \in \mathit{ucl}_{\mathcal{T}}(\mathcal{A})$, while $A(c) \notin \mathcal{J}$ implies $A(c) \notin \mathcal{A}'$. Thus $\{A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. On the other hand, $A(c) \in \mathcal{J}'$ and $\mathcal{J}' \models \mathcal{N}$ imply that $\{A(c)\} \cup \mathcal{N} \not\models \bot$, which yields a contradiction.

(ii) $A(c) \in \mathcal{J} \setminus \mathcal{I}$ and $A(c) \notin \mathcal{J}'$: From $A(c) \notin \mathcal{J}'$ and $\mathcal{J}' \models \mathcal{N}$ we imply that $\mathcal{N} \not\models A(c)$. From $A(c) \in \mathcal{J}$ and $A(c) \notin \mathcal{I}$ we imply that $\{A(c)\} \cup \mathcal{A} \models_{\mathcal{T}} \bot$ and therefore $\neg A(c) \in \mathit{fcl}_{\mathcal{T}}\mathcal{A}$. Since $A(c) \in \mathcal{J}$ and $\neg A(c) \notin \texttt{AlignAlg}(\mathcal{K}, \mathcal{N})$, thus, $\{\neg A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$ and, therefore, we obtain $\mathcal{N} \models A(c)$, which yields a contradiction.

Thus, $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ and we conclude the proof. □

Let $\mathcal{T}$ be a *DL-Lite*$_{\mathcal{R}}$ TBox and $S^n_{\mathcal{T}}$ a sequence $\{f_1, \ldots, f_n, L\}$, where $f_1, \ldots, f_n$ are ground atoms and L is a ground literal, such that there is a sequence of TBox assertions $\alpha_1, \ldots, \alpha_n$ from $\mathit{cl}(\mathcal{T})$, where each α_i is *not* of the form $\exists R \sqsubseteq \exists R$, and $f_i \to f_{i+1}$ is an instantiation of the first-order interpretation of α_i for $1 \le i \le n$. Note that α_n can be either a PI (if L is positive) or NI (if L is negative), while all the other α_is are PIs. Then,

$$\mathit{root}_{\mathcal{T}}(C(a)) = \bigcup_{\mathcal{S}^n_{\mathcal{T}}:\ n \in \mathbb{N},\ L = C(a)} \mathcal{S}_T, \qquad \mathit{root}_{\mathcal{T}}(R(a,b)) = \bigcup_{\substack{\mathcal{S}^n_{\mathcal{T}}:\ n \in \mathbb{N},\ L = R(a,d) \text{ or} \\ L = R(d,b) \text{ for some } d \in \Delta}} \mathcal{S}_T.$$

If $\mathcal{I}$ is an interpretation, then $\mathit{root}^I_{\mathcal{T}}(C(a))$ denotes a restriction of $\mathit{root}_{\mathcal{T}}(C(a))$, where each $\mathcal{S}_{\mathcal{T}}$ in the union satisfies: $\mathcal{S}_T \subseteq \mathcal{I}$. Note that in the following whenever we write $\mathcal{I} \setminus \mathit{root}_{\mathcal{T}}(g)$ for an MA g and a model $\mathcal{I}$, we always mean $\mathcal{I} \setminus \mathit{root}^I_{\mathcal{T}}(g)$.

For example, consider the TBox $\mathcal{T} = \{A \sqsubseteq \exists R\}$, and $C(a) = \exists R(a)$, then $A(a) \in \mathit{root}_{\mathcal{T}}(\exists R(a))$. Consider another TBox $\mathcal{T} = \{A \sqsubseteq \neg B\}$ and $C(a) = \neg B(a)$, then $A(a) \in \mathit{root}_{\mathcal{T}}(\neg B(a))$. Now we are ready to proceed to the proof of Theorem 12.

Proof of Theorem 12. Polynomiality of $\texttt{AlignAlg}$ in $|\mathcal{K} \cup \mathcal{N}|$ follows from worst case quadratic size of $\mathit{fcl}_{\mathcal{T}}(\mathcal{A})$ in $|\mathcal{K}|$ and polynomiality in $|\mathcal{K} \cup \mathcal{N}|$ of the test $\{\phi\} \cup \mathcal{N} \not\models_{\mathcal{T}} \bot$. Let $\mathcal{M}$ denote the set of models $\mathcal{K} \diamond \mathcal{N}$. The result of alignment of $\mathcal{K}$ and $\mathcal{N}$ is the ABox $\mathcal{A}'' = \texttt{AlignAlg}(\mathcal{K}, \mathcal{N})$, and $\mathcal{A}' = \mathcal{A}'' \cup \mathcal{N}$, thus, $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$. We now prove that $\mathcal{K}'$ is a representation of $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ by showing two inclusions: $\mathcal{M} \subseteq \mathit{Mod}(\mathcal{K}')$ and $\mathit{Mod}(\mathcal{K}') \subseteq \mathcal{M}$.

$\mathcal{M} \subseteq \mathit{Mod}(\mathcal{K}')$: Let $\mathcal{J} \in \mathcal{M}$, we will show $\mathcal{J} \in \mathit{Mod}(\mathcal{K}')$, i.e., $\mathcal{J} \in \mathit{Mod}(\mathcal{T})$ and $\mathcal{J} \in \mathit{Mod}(\mathcal{A}')$. By definition of $\mathcal{L}^a_{\subseteq}$, $\mathcal{J} \in \mathcal{M}$ implies $\mathcal{J} \in \mathit{Mod}(\mathcal{T})$. Assume $\mathcal{J} \notin \mathit{Mod}(\mathcal{A}')$.

Since $\mathcal{J} \in \mathcal{M}$ we have $\mathcal{J} \models \mathcal{N}$. Since also $\mathcal{J} \notin \mathit{Mod}(\mathcal{A}')$ we have $\mathcal{J} \not\models \mathcal{A}''$. Thus, there is an MA $g \in \mathcal{A}''$ such that $\mathcal{J} \not\models g$, where the assertion g can either be a positive MA, or a negative one $g = \neg h$, where h is a positive MA. From $\mathcal{J} \not\models g$ we will deduce a contradiction by showing $\mathcal{J} \notin \mathcal{M}$, that is, by showing that for every $\mathcal{I} \in \mathit{Mod}(\mathcal{K})$ there is $\mathcal{J}' \in \mathit{Mod}(\mathcal{T}) \cap \mathit{Mod}(\mathcal{N})$ such that

$$\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}. \tag{6}$$

Let g be a positive MA. Consider an arbitrary $\mathcal{I} \in \mathit{Mod}(\mathcal{K})$. Clearly, $\mathcal{I} \models g$ (since $\mathcal{I} \models_{\mathcal{T}} \mathcal{A}''$ and $g \in \mathcal{A}''$). Now let $\mathcal{J}' := \mathcal{J} \cup \mathcal{I}[g]$. Clearly, such $\mathcal{I}[g]$ exists while it may be not unique. If $\mathcal{I}[g]$ is not unique, then any such $\mathcal{I}[g]$ can be used in the construction of $\mathcal{J}'$. Observe that $\mathcal{J}' \models \mathcal{N}$ and $\mathcal{J}' \models \mathcal{T}$. The former holds since $\mathcal{J} \models \mathcal{N}$. We will show the latter entailment using the following property of *DL-Lite*$_{\mathcal{R}}$:

Proposition 15. *Let $\mathcal{T}$ be a DL-Lite$_{\mathcal{R}}$ TBox, and $\mathcal{I}_1$, $\mathcal{I}_2$ be models of $\mathcal{T}$. Then $\mathcal{I}_1 \cup \mathcal{I}_2 \models \mathcal{T}$ if and only if for every $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$, it holds that $\{f_1, f_2\} \not\models_{\mathcal{T}} \bot$.*

Assume $\mathcal{J}' \not\models \mathcal{T}$. Since $\mathcal{J} \models \mathcal{T}$, $\mathcal{I}[g] \models \mathcal{T}$, and due to Proposition 15, there are $f_1 \in \mathcal{J}$ and $f_2 \in \mathcal{I}[g]$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. Clearly, $f_1 \notin \mathcal{I}[g]$ and due to $\mathcal{I}[g] \subseteq \mathcal{I}$ we have that $f_1 \notin \mathcal{I}$. Now consider an interpretation $\mathcal{J}_1 = \mathcal{J} \setminus \mathit{root}_{\mathcal{T}}(f_1)$. We will show that $\mathcal{J}_1 \models (\mathcal{N} \cup \mathcal{T})$ and $\mathcal{I} \ominus \mathcal{J}_1 \subsetneq \mathcal{I} \ominus \mathcal{J}$, thus $\mathcal{J} \notin \mathcal{M}$, which will give us a contradiction. The entailment $\mathcal{J}_1 \models \mathcal{T}$ trivially holds. To see that $\mathcal{J}_1 \models \mathcal{N}$ observe that $\mathcal{N} \not\models_{\mathcal{T}} f_1$. Indeed, if $\mathcal{N} \models_{\mathcal{T}} f_1$, then from $g \models_{\mathcal{T}} f_2$ and $\{f_1, f_2\} \models_{\mathcal{T}} \bot$ we can derive that $\mathcal{N} \cup \{g\} \models \bot$ which contradicts with the assumption that $g \in \mathcal{A}''$ (such g should have been dropped by `AlignAlg` from $\mathcal{A}''$, see Line 4 in Algorithm 1). Using the definition of $\ominus$ and the fact that $\mathcal{J}_1 \subseteq \mathcal{J}$ one can easily verify that $\mathcal{I} \ominus \mathcal{J}_1 \subseteq \mathcal{I} \ominus \mathcal{J}$. Inequality $\mathcal{I} \ominus \mathcal{J}_1 \neq \mathcal{I} \ominus \mathcal{J}$ follows from the fact that $f_1 \notin \mathcal{I}$, $f_1 \notin \mathcal{J}_1$, and $f_1 \in \mathcal{J}$. This finishes the proof of $\mathcal{J}' \models \mathcal{T}$. It remains to show that the Equation 6 holds for the constructed $\mathcal{J}'$. Since $\mathcal{I}[g] \subseteq \mathcal{I}$ one can apply the definition of $\ominus$ to conclude that $\mathcal{I} \ominus \mathcal{J}' = \mathcal{I} \ominus (\mathcal{J} \cup \mathcal{I}[g]) \subseteq \mathcal{I} \ominus \mathcal{J}$ indeed holds, while $\mathcal{I} \ominus \mathcal{J}' \neq \mathcal{I} \ominus \mathcal{J}$ follows from the fact that $g \in \mathcal{I}$, $g \in \mathcal{J}'$, and $g \notin \mathcal{J}$. Thus, $\mathcal{J} \notin \mathcal{M}$ and we obtain a contradiction.

Let $g = \neg h$ be a negative MA. Consider an arbitrary $\mathcal{I} \in \mathit{Mod}(\mathcal{K})$. Clearly, $\mathcal{J} \models h$ and $\mathcal{I} \not\models h$ (since $\mathcal{I} \models_{\mathcal{T}} \mathcal{A}''$ and $\neg h \in \mathcal{A}''$). Now let $\mathcal{J}' := \mathcal{J} \setminus \mathit{root}_{\mathcal{T}}(h)$. Observe that $\mathcal{J}' \models \mathcal{N}$ and $\mathcal{J}' \models \mathcal{T}$. The former holds since $\neg h \in \mathcal{A}''$ and consequently $\{\neg h\} \cup \mathcal{N} \not\models_{\mathcal{T}} \bot$, thus, $\mathit{root}_{\mathcal{T}}(h) \cap \mathcal{N} = \emptyset$. The latter holds due to the following proposition.

Proposition 16. *Let $\mathcal{T}$ be a DL-Lite$_{\mathcal{R}}$ TBox and $\mathcal{I} \models \mathcal{T}$. Then $\mathcal{I} \setminus \mathit{root}_{\mathcal{T}}(g) \models \mathcal{T}$ for every general MA g.*

The proof of $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ is analogous to the one of $\mathcal{I} \ominus \mathcal{J}_1 \subsetneq \mathcal{I} \ominus \mathcal{J}$ above. We conclude that Equation 6 holds, $\mathcal{J} \notin \mathcal{M}$, and obtain a contradiction.

$\mathcal{M} \supseteq \mathit{Mod}(\mathcal{K}')$: Let $\mathcal{J} \models \mathcal{K}'$. To prove that $\mathcal{J} \in \mathcal{M}$ we need to show that $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$ and there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$. The former follows from the fact that $\mathcal{J} \models \mathcal{K}'$, while the latter from Lemma 13. Thus, $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ and therefore $\mathcal{M} \supseteq \mathit{Mod}(\mathcal{K}')$ which concludes the proof. □

In Lemma 13 we showed how to contract $\mathcal{I}$ given $\mathcal{J}$ such that $\mathcal{J}$ is in the evolution of $\mathcal{I}$ under $\mathcal{L}^a_{\subseteq}$ and finitely $\mathit{dist}^a_{\subseteq}$-distanced from $\mathcal{I}$. This property was shown for *DL-Lite*$^{pr}_{\mathcal{R}}$, a fragment of *DL-Lite*$_{\mathcal{R}}$. We now show that every model $\mathcal{I}$ of a *DL-Lite*$_{\mathcal{R}}$ KB $\mathcal{K}$ can be evolved under $\mathcal{L}^a_{\subseteq}$ in a model $\mathcal{J}$ that is finitely $\mathit{dist}^a_{\subseteq}$-distanced (and consequently $\mathit{dist}^a_{\#}$-distanced) from $\mathcal{I}$.

Proposition 17. *Let $\mathcal{K}$ be a DL-Lite$_{\mathcal{R}}$ KB, $\mathcal{K}$ and $\mathcal{N}$ an evolution setting, and $\mathcal{I}$ a model of $\mathcal{K}$. Then, there is a model $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ such that $\mathit{dist}^a_{\subseteq}(\mathcal{I}, \mathcal{J})$ is finite.*

Relationships Between Atom-Based Semantics. The next theorem shows that in *DL-Lite*$^{pr}_{\mathcal{R}}$ all four atom-based MBAs coincide. We depict these in Figure 2 using dashed arrows, e.g., between $\mathcal{L}^a_{\subseteq}$ and $\mathcal{G}^a_{\#}$. The figure should be read as follows: there is an oriented path with solid or dashed arrows (a sequence of such arrows) between any two semantics $\mathcal{S}_1$ and $\mathcal{S}_2$ if and only if $(\mathcal{S}_1 \preccurlyeq_{sem} \mathcal{S}_2)(\mathit{DL\text{-}Lite}^{pr}_{\mathcal{R}})$.

Theorem 18. *For DL-Lite$^{pr}_{\mathcal{R}}$ we have that $\mathcal{L}^a_{\#} \equiv_{sem} \mathcal{L}^a_{\subseteq} \equiv_{sem} \mathcal{G}^a_{\#} \equiv_{sem} \mathcal{G}^a_{\subseteq}$.*

Proof. Due to Theorem 6 and transitivity of $\preccurlyeq_{sem}$, it suffices to show two relations: $\mathcal{L}^a_{\#} \preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$ and $\mathcal{L}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^a_{\#}$. We start with the former one.

$\mathcal{L}^a_{\#} \preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$: Consider a model $\mathcal{J}_0 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\#}$ for an evolution setting $\mathcal{K}$ and $\mathcal{N}$. This yields that there is a model $\mathcal{I}_0 \models \mathcal{K}$ such that $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\#}$. Then, if we consider a model $\mathcal{I}'_0$ built as in Equation 5, then we will have that $\mathit{dist}^a_{\#}(\mathcal{I}_0, \mathcal{J}_0) \leq \mathit{dist}^a_{\#}(\mathcal{I}'_0, \mathcal{J}_0)$. The latter distance is finite by the construction of $\mathcal{I}'_0$, so is the former distance, and therefore $\mathcal{L}^a_{\#} \preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$ (see Theorem 6). The only thing we need to show is that $\mathcal{I}'_0 \models \mathcal{K}$. This can be shown similarly to the prove of Lemma 13 using the following proposition instead of Proposition 14.

Proposition 19. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a DL-Lite$^{pr}_{\mathcal{R}}$ KB and $\mathcal{K}$, $\mathcal{N}$ an evolution setting. Let $\mathcal{S}$ be $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\#}$. For every MA g of the form $R(a,b)$ or $\exists R(a)$, if $\mathcal{A} \models_{\mathcal{T}} g$, then for every $\mathcal{J} \in \mathcal{S}$ it holds $\mathcal{I} \models g$.*

$\mathcal{L}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^a_{\#}$. Consider an evolution setting $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ where $\mathcal{K}$ is in *DL-Lite*$^{pr}_{\mathcal{R}}$. Let $\mathcal{M}_L$ and $\mathcal{M}_G$ be the results of evolution $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ and $\mathcal{G}^a_{\#}$, respectively. Consider a model $\mathcal{J} \in \mathcal{M}_L$. Now we will show that $\mathcal{J} \in \mathcal{M}_G$, that is, that there is a model $\mathcal{I} \models \mathcal{K}$ such that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{K}$ under $\mathcal{G}^a_{\#}$, i.e., for every $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I}' \models \mathcal{K}$ it *does not* hold that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I} \ominus \mathcal{J}|$. Consider $\mathcal{I}$ as in Equation 5. Due to Lemma 13, $\mathcal{I} \models \mathcal{K}$. Assume there are $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I}' \models \mathcal{K}$ such that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I} \ominus \mathcal{J}|$. Since the set $\mathcal{I} \ominus \mathcal{J}$ is at most countable, $\mathcal{I}' \ominus \mathcal{J}'$ is finite, and, therefore, there exists an atom $A(c) \in (\mathcal{I} \ominus \mathcal{J}) \setminus (\mathcal{I}' \ominus \mathcal{J}')$. We have two cases:

(i) $A(c) \in \mathcal{I} \setminus \mathcal{J}$: From the definition of $\mathcal{I}$ these two conditions imply that $A(c) \in \mathit{ucl}_{\mathcal{T}}(\mathcal{A})$. Observe that $\mathit{ucl}_{\mathcal{T}}(\mathcal{A}) \subseteq \mathit{fcl}_{\mathcal{T}}(\mathcal{A})$, and consequently $A(c) \in \mathit{fcl}_{\mathcal{T}}(\mathcal{A})$. Due to Theorem 12 the inclusion $\mathcal{J} \in \mathcal{M}_L$ implies that $\mathcal{J}$ is a model of $\mathcal{K}' = \mathcal{T} \cup \mathcal{A}'$, where $\mathcal{A}' = \texttt{AlignAlg}(\mathcal{T} \cup \mathcal{A}, \mathcal{N}) \cup \mathcal{N}$. Due to $A(c) \notin \mathcal{J}$ we get $A(c) \notin \texttt{AlignAlg}(\mathcal{T} \cup \mathcal{A}, \mathcal{N})$. From the last condition and $A(c) \in \mathit{fcl}_{\mathcal{T}}(\mathcal{A})$, we obtain $\{A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$ (this follows from the definition of `AlignAlg`). This entailment together with $\mathcal{J}' \models \mathcal{N}$ implies that $A(c) \notin \mathcal{J}'$. From $A(c) \notin \mathcal{J}'$ and $A(c) \notin \mathcal{I}' \ominus \mathcal{J}'$ we get $A(c) \notin \mathcal{I}'$. Finally, since $A(c) \in \mathit{fcl}_{\mathcal{T}}(\mathcal{A})$ and $\mathcal{I}' \models \mathit{fcl}_{\mathcal{T}}(\mathcal{A})$, we have that $A(c) \in \mathcal{I}'$, which yields a contradiction.

(ii) $A(c) \in \mathcal{J} \setminus \mathcal{I}$: By the definition of $\mathcal{I}$ these two conditions imply $A(c) \in \mathcal{J}^{\mathcal{A}}$ and also $\{A(c)\} \cup \mathit{fcl}_{\mathcal{T}}(\mathcal{A}) \models_{\mathcal{T}} \bot$, which is equivalent to $\neg A(c) \in \mathit{fcl}_{\mathcal{T}}(\mathcal{A})$. Thus, $A(c) \notin \mathcal{I}'$ (otherwise $\mathcal{I}'$ would not be a model of $\mathcal{K}$). Recall that $A(c) \notin \mathcal{I}' \ominus \mathcal{J}'$, and, therefore, $A(c) \notin \mathcal{J}'$. We obtain that $\mathcal{J}' \not\models A(c)$. Since $A(c) \in \mathcal{J}$, we have $\neg A(c) \notin \mathcal{A}'$, that is, $\{\neg A(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. Now combining the last entailment with $\mathcal{J}' \models \mathcal{N}$ we conclude $\mathcal{J}' \not\models \neg A(c)$, which contradicts $\mathcal{J}' \not\models A(c)$.

Thus, $\mathcal{J} \in \mathcal{M}_G$ and consequently $\mathcal{L}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^a_{\#}$. □

Theorems 12 and 18 imply that in *DL-Lite*$^{pr}_{\mathcal{R}}$ one can use `AlignAlg` to compute (a representation of) evolution under *all* atom based MBAs, and the computation time is polynomial in the size of both $\mathcal{K}$ and $\mathcal{N}$.

4.2. Capturing Symbol-Based Evolution

On Relationships Between Semantics. It turns out that for symbol-based MBAs the local semantics based on cardinality and on set inclusion coincide, as well the global ones, while local semantics are not subsumed by the global ones. Moreover, all atom-based MBAs are subsumed by the global symbol-based semantics, while the contrary does not hold. Thus, for *DL-Lite*$^{pr}_{\mathcal{R}}$ we essentially have three different evolution semantics: atom-based, local symbol-based, and global symbol-based. We sum this up in the following theorem and illustrate it in Figure 2.

Theorem 20. *The following relations holds for DL-Lite$^{pr}_{\mathcal{R}}$:*

(i) $\mathcal{L}^s_{\subseteq} \equiv_{sem} \mathcal{L}^s_{\#}$, *and* $\mathcal{G}^s_{\subseteq} \equiv_{sem} \mathcal{G}^s_{\#}$, *while* $\mathcal{L}^s_{\subseteq} \not\preccurlyeq_{sem} \mathcal{G}^s_{\#}$;

(ii) $\mathcal{L}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^s_{\#}$, *while* $\mathcal{G}^s_{\#} \not\preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$.

To prove the theorem we need extra notation. With $\mathcal{A} \parallel_{\mathcal{T}} \phi$ we denote the fact that neither $\mathcal{A} \models_{\mathcal{T}} \phi$ nor $\mathcal{A} \models_{\mathcal{T}} \neg\phi$ holds. The definition of $\mathcal{K} \parallel \phi$ is analogous. Observe that for every KB $\mathcal{K}$ and atom $A(c)$, there are three possible relations between them: $\mathcal{K} \models A(c)$, or $\mathcal{K} \models \neg A(c)$, or $\mathcal{K} \parallel A(c)$. For a given $\mathcal{K}$ and $\mathcal{N}$ (together with $\mathcal{T}$) these three relations give nine combinations, which are presented in Figure 3. We call each combination a *type of* $A(c)$ (wrt $\mathcal{K}$ and $\mathcal{N}$) and use them in the proofs of Theorems 20 and 22.

Proof of Theorem 20.

(T1)	$\mathcal{N} \models_{\mathcal{T}} A(c)$ and $\mathcal{K} \models A(c)$;		*(T6)*	$\mathcal{N} \models_{\mathcal{T}} \neg A(c)$ and $\mathcal{K} \parallel A(c)$;	
(T2)	$\mathcal{N} \models_{\mathcal{T}} \neg A(c)$ and $\mathcal{K} \models \neg A(c)$;		*(T7)*	$\mathcal{N} \parallel_{\mathcal{T}} A(c)$ and $\mathcal{K} \parallel A(c)$;	
(T3)	$\mathcal{N} \models_{\mathcal{T}} A(c)$ and $\mathcal{K} \models \neg A(c)$;		*(T8)*	$\mathcal{N} \parallel_{\mathcal{T}} A(c)$ and $\mathcal{K} \models A(c)$;	
(T4)	$\mathcal{N} \models_{\mathcal{T}} \neg A(c)$ and $\mathcal{K} \models A(c)$;		*(T9)*	$\mathcal{N} \parallel_{\mathcal{T}} A(c)$ and $\mathcal{K} \models \neg A(c)$.	
(T5)	$\mathcal{N} \models_{\mathcal{T}} A(c)$ and $\mathcal{K} \parallel A(c)$;				

Figure 3: Classification of atoms $A(c)$ wrt $\mathcal{K}$ and $\mathcal{N}$

$\mathcal{G}^s_{\#} \equiv_{sem} \mathcal{G}^s_{\subseteq}$: Due to Theorem 6 it suffices to show $\mathcal{G}^s_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^s_{\#}$. Let $\mathcal{M}_{\subseteq}$ and $\mathcal{M}_{\#}$ be the results of evolution $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^s_{\subseteq}$ and $\mathcal{G}^s_{\#}$, respectively. Consider a model $\mathcal{J}_0 \in \mathcal{M}_{\subseteq}$, we show that $\mathcal{J}_0 \in \mathcal{M}_{\#}$. By definition of $\mathcal{G}^s_{\subseteq}$ semantics, there is a model $\mathcal{I}_0 \in Mod(\mathcal{K})$, such that for every pair of models $\mathcal{I} \in Mod(\mathcal{K})$ and $\mathcal{J} \in Mod(\mathcal{T} \cup \mathcal{N})$ it *does not* hold that $dist^s_{\subseteq}(\mathcal{I}, \mathcal{J}) \subsetneq dist^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$. Suppose that $\mathcal{J}_0 \notin \mathcal{M}_{\#}$, that is, for each model $\mathcal{I}' \in Mod(\mathcal{K})$ there are models $\mathcal{I} \in Mod(\mathcal{K})$ and $\mathcal{J} \in Mod(\mathcal{T} \cup \mathcal{N})$ such that $|dist^s_{\subseteq}(\mathcal{I}, \mathcal{J})| \lneq |dist^s_{\subseteq}(\mathcal{I}', \mathcal{J}_0)|$. In particular it holds when $\mathcal{I}' = \mathcal{I}_0$. This implies that there is an element in the signature of $\mathcal{K} \cup \mathcal{N}$ with the same interpretation in $\mathcal{I}$ and $\mathcal{J}$, and different interpretations in $\mathcal{I}_0$ and $\mathcal{J}_0$. If this element is a concept A, then $A^{\mathcal{I}} = A^{\mathcal{J}}$ and $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$ (the case of a role is analogous). Thus, there is an atom $A(c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$ for some $c \in \Delta$. We now exhibit models $\mathcal{I}_1 \models \mathcal{K}$ and $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$ s.t.

$$dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \subsetneq dist^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0), \tag{7}$$

which contradicts the assumption $\mathcal{J}_0 \in \mathcal{M}_{\subseteq}$. To achieve this, we build $\mathcal{I}_1$ and $\mathcal{J}_1$ from $\mathcal{I}_0$ and $\mathcal{J}_0$, respectively, in a way that they agree on the interpretation of A. The construction of these models depends on the type of each $A(c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$ wrt $\mathcal{K}, \mathcal{N}$ (Figure 3). Observe that $A(c)$ *cannot* be of type *(T1)-(T4)*. Indeed, if $A(c)$ is of type *(T1)* or *(T2)*, then $A(c) \notin \mathcal{I}_0 \ominus \mathcal{J}_0$. If $A(c)$ is of type *(T3)* or *(T4)*, then $A(c) \in \mathcal{I} \ominus \mathcal{J}$. Both cases contradict with $A(c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I} \ominus \mathcal{J})$. For $A(c)$ of type *(T5)-(T7)*, we construct $\mathcal{I}_1$ from $\mathcal{I}_0$ as follows:

$$\mathcal{I}_1 := \bigcup_{\substack{A(c) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\ \mathcal{J}_0 \models A(c) \\ A(c) \text{ of type } (T5) \text{ or } (T7)}} \Big((\mathcal{I}_0 \setminus root_{\mathcal{T}}(\neg A(c))) \cup \mathcal{J}_0[A(c)] \Big) \setminus \bigcup_{\substack{A(c) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\ \mathcal{J}_0 \models \neg A(c) \\ A(c) \text{ of type } (T6)}} root_{\mathcal{T}}(A(c)). \tag{8}$$

We recall that $\mathcal{J}_0[A(c)]$ is the minimal submodel of $\mathcal{J}_0$ containing $A(c)$ and satisfying $\mathcal{T}$. Due to Proposition 16 and the following Proposition 21 we have $\mathcal{I}_1 \models \mathcal{T}$.

Proposition 21. *Let $\mathcal{T}$ be a DL-Lite$^{pr}_{\mathcal{R}}$ TBox, $\mathcal{I}$, $\mathcal{J}$ models of $\mathcal{T}$, and $A(c)$ an atom. Then the interpretation $(\mathcal{I} \setminus root_{\mathcal{T}}(\neg A(c))) \cup \mathcal{J}[A(c)]$ is a model of $\mathcal{T}$.*

Recall that $A(c)$ is of type *(T5)-(T6)* and therefore $\mathcal{K} \parallel A(c)$. Moreover, one can show that due to the fact that $\mathcal{K}$ is in *DL-Lite*$^{pr}_{\mathcal{R}}$ and $\mathcal{J}_0 \models \neg A(c)$, each subtracted set $root_{\mathcal{T}}(A(c))$ contains only unary atoms of the form $A'(c)$. Combining these two observation we conclude that $\mathcal{K} \parallel A'(c)$ and therefore $A'(c) \notin \mathcal{A}$. Thus, $\mathcal{I}_1 \models \mathcal{A}$. For $A(c)$ of type *(T7)-(T9)* we construct $\mathcal{J}_1$ from $\mathcal{J}_0$ as follows:

$$\mathcal{J}_1 := \bigcup_{\substack{A(c) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\ \mathcal{I}_0 \models A(c) \\ A(c) \text{ of type } (T8) \text{ or } (T7)}} \Big((\mathcal{J}_0 \setminus root_{\mathcal{T}}(\neg A(c))) \cup \mathcal{I}_0[A(c)] \Big) \setminus \bigcup_{\substack{A(c) \in \mathcal{I}_0 \ominus \mathcal{J}_0,\ \mathcal{I}_0 \models \neg A(c) \\ A(c) \text{ of type } (T9)}} root_{\mathcal{T}}(A(c)). \tag{9}$$

One can show that $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$ analogously to the proof of $\mathcal{I}_1 \models \mathcal{T} \cup \mathcal{A}$ above. Observe that by construction of $\mathcal{I}_1$ and $\mathcal{J}_1$, we have $A^{\mathcal{I}_1} = A^{\mathcal{J}_1}$ and $dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) \subseteq dist^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$. Finally, the former equality gives that $A \notin dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1)$, which together with $A \in dist^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$ implies Equation 7 and concludes the proof.

Algorithm 2: Algorithm $\texttt{SymAlg}(\mathcal{T} \cup \mathcal{A}, \mathcal{N}, \Pi)$ to compute $(\mathcal{T} \cup \mathcal{A}) \diamond \mathcal{N}$ under $\mathcal{G}^s_\subseteq$ and $\mathcal{G}^s_\#$ semantics and minimal sound approximation under $\mathcal{L}^s_\subseteq$ and $\mathcal{L}^s_\#$ semantics (Theorems 22 and 25)

INPUT : evolution setting $(\mathcal{T}, \mathcal{A})$ and $\mathcal{N}$ with $\mathcal{T}$ in *DL-Lite*$^{pr}_\mathcal{R}$, and a property Π of MAs
OUTPUT: a set $\mathcal{A}' \subseteq fcl_\mathcal{T}(\mathcal{A}) \cup fcl_\mathcal{T}(\mathcal{N})$ of ABox assertions

1 $\mathcal{A}' := \emptyset$; $\mathcal{S}_1 := \texttt{AlignAlg}(\mathcal{T} \cup \mathcal{A}, \mathcal{N})$; $\mathcal{S}_2 := fcl_\mathcal{T}(\mathcal{N})$;
2 **repeat**
3 | **choose some** $\phi \in \mathcal{S}_2$; $\mathcal{S}_2 := \mathcal{S}_2 \setminus \{\phi\}$;
4 | **if** $\Pi(\phi) = \textit{true}$ **then** $\mathcal{S}_1 := \mathcal{S}_1 \setminus \{\phi' \in fcl_\mathcal{T}(\mathcal{A}) \mid \phi$ and ϕ' have the same concept name$\}$
5 **until** $\mathcal{S}_2 = \emptyset$;
6 $\mathcal{A}' := \mathcal{S}_1 \cup \mathcal{N}$

$\mathcal{L}^s_\# \equiv_{sem} \mathcal{L}^s_\subseteq$: Due to Theorem 6 it suffices to show $\mathcal{L}^s_\subseteq \preccurlyeq_{sem} \mathcal{L}^s_\#$. This can be done similarly to the case of $\mathcal{G}^s_\# \preccurlyeq_{sem} \mathcal{G}^s_\subseteq$, by proving $dist^s_\subseteq(\mathcal{I}_0, \mathcal{J}_1) \subsetneq dist^s_\subseteq(\mathcal{I}_0, \mathcal{J}_0)$ with $\mathcal{J}_1$ for $A(c)$ of types *(T7)-(T9)*.

$\mathcal{L}^a_\subseteq \preccurlyeq_{sem} \mathcal{G}^s_\#$: Consider a model $\mathcal{J}$ of $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$, we show that $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^s_\#$. Due to Theorem 12, $\mathcal{J}$ is a model of $\mathcal{K}'$ as in Equation 4. Let $\mathcal{I}$ be an interpretation built as in Equation 5, Due to Lemma 13, we obtain $\mathcal{I} \models \mathcal{K}$. Suppose that $\mathcal{J} \not\models \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^s_\#$, that is, there exists a model $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ such that $|dist^s_\subseteq(Mod(\mathcal{K}), \mathcal{J}')| \lneq |dist^s_\subseteq(Mod(\mathcal{K}), \mathcal{J})|$. By definition of the distance between a set of interpretations and an interpretation, there exists a model $\mathcal{I}' \models \mathcal{K}$ such that $|dist^s_\subseteq(\mathcal{I}', \mathcal{J}')| \lneq |dist^s_\subseteq(\mathcal{I}, \mathcal{J})|$. This implies that there is an element in the signature of $\mathcal{K} \cup \mathcal{N}$ with the same interpretation in $\mathcal{I}'$ and $\mathcal{J}'$, while it is interpreted differently in $\mathcal{I}$ and $\mathcal{J}$. We consider the case when this element is a concept A, i.e., $A^\mathcal{I} \neq A^\mathcal{J}$ and $A^{\mathcal{I}'} = A^{\mathcal{J}'}$, the case of a role is analogous. From $A^\mathcal{I} \neq A^\mathcal{J}$ and Equation 5 we imply that there is an atom $A(c)$ that is either in $\mathcal{J}^\mathcal{A}$ or in $ucl_\mathcal{T}(A)$. If $A(c) \in \mathcal{J}^\mathcal{A}$, then $\neg A(c) \in fcl_\mathcal{T}(\mathcal{A})$, and, since $\mathcal{J} \not\models \neg A(c)$, the literal $\neg A(c)$ was deleted from $fcl_\mathcal{T}(\mathcal{A})$ while building $\mathcal{A}'$ (see Algorithm 1), i.e., $\{\neg A(c)\} \cup \mathcal{N} \models_\mathcal{T} \bot$. From this entailment and $\mathcal{J}' \models \mathcal{N}$ we conclude that $\mathcal{J}' \not\models \neg A(c)$ and consequently $\mathcal{I}' \not\models \neg A(c)$ (since $A^{\mathcal{I}_1} = A^{\mathcal{J}_1}$). We obtain a contradiction with $\mathcal{I}_1 \models fcl_\mathcal{T}(\mathcal{A})$. If $A(c) \in ucl_\mathcal{T}(A)$, then $A(c) \in fcl_\mathcal{T}(\mathcal{A})$ and $\mathcal{I}' \models A(c)$. Due to $A^{\mathcal{I}'} = A^{\mathcal{J}'}$, we have that $\mathcal{J}' \models A(c)$. On the other hand, since $\mathcal{I} \models ucl_\mathcal{T}(\mathcal{A})$ and $A^\mathcal{I} \neq A^\mathcal{J}$, $\mathcal{J} \not\models A(c)$, which implies $\{A(c)\} \cup \mathcal{N} \models_\mathcal{T} \bot$. Recall that $\mathcal{J}' \models \mathcal{N}$, thus, $\mathcal{J}' \not\models A(c)$ and we obtain a contradiction.

$\mathcal{L}^s_\subseteq \not\preccurlyeq_{sem} \mathcal{G}^s_\#$ and $\mathcal{G}^s_\# \not\preccurlyeq_{sem} \mathcal{L}^a_\subseteq$: Consider evolution of the KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, where $\mathcal{T} = \{B \sqsubseteq \neg C\}$ and $\mathcal{A} = \{A(c), B(a), B(d)\}$ with $\mathcal{N} = \{A(e), C(d)\}$. Consider two models of $\mathcal{T} \cup \mathcal{N}$: $\mathcal{J}_1 = \{A(e), C(d)\}$ and $\mathcal{J}_2 = \{A(c), A(e), C(d)\}$. To conclude the proof observe that $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^s_\subseteq$, but $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^s_\#$, and $\mathcal{J}_2 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^s_\#$, but $\mathcal{J}_2 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$. □

As a corollary of Theorem 20, the approach to compute representations of evolutions $\mathcal{K} \diamond \mathcal{N}$ presented in Theorem 12 does not work for computing $\mathcal{K} \diamond \mathcal{N}$ under *any* of the symbol-based MBAs. We now introduce an algorithm that we will use to compute and approximate evolution under symbol-based MBAs.

Capturing Global Semantics. Consider the algorithm $\texttt{SymAlg}$ in Algorithm 2. It takes an evolution setting $\mathcal{T} \cup \mathcal{A}$, $\mathcal{N}$, and a unary property Π of assertions as input. Then, for every atom ϕ in $\mathcal{N}$ it checks whether ϕ satisfies Π (Line 4). If it is the case, $\texttt{SymAlg}$ deletes from $\texttt{AlignAlg}(\mathcal{T} \cup \mathcal{A}, \mathcal{N})$ all literals ϕ' that share a concept name with ϕ. Local and global semantics have their own Π: $\Pi_\mathcal{G}$ and $\Pi_\mathcal{L}$, respectively. The property $\Pi_\mathcal{G}(\phi)$ checks whether ϕ of $\mathcal{N}$ $\mathcal{T}$-contradicts $\mathcal{A}$, i.e.,

$$\Pi_\mathcal{G}(\phi) \text{ is true if and only if } \neg\phi \in fcl_\mathcal{T}(\mathcal{A}).$$

Intuitively, to compute $\mathcal{K} \diamond \mathcal{N}$ under global symbol-based semantics we should trace all assertions of the form $B(c)$ or $\neg B(c)$ entailed by $\mathcal{A}$ that should be deleted from (the $\mathcal{T}$-closure of) $\mathcal{A}$ due to $\mathcal{N}$. For such B's the change of interpretation is inevitable, i.e., if in some model $\mathcal{I}$ of $\mathcal{K}$ we had $b \in B^{\mathcal{I}}$, then in every $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ we have $b \notin B^{\mathcal{I}}$. Since symbol-based semantics trace changes on symbols only, and the interpretation of the symbol B should be changed, one should drop from (the $\mathcal{T}$-closure of) $\mathcal{A}$ all the assertions over the symbol B, that is, of the form $B(d)$ and $\neg B(d)$ for some d. The following theorem shows correctness of this algorithm.

Theorem 22. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ be an evolution setting, and $\mathcal{T}$ in DL-Lite$^{pr}_{\mathcal{R}}$. Then*

$$\mathcal{K}' = \mathcal{T} \cup \texttt{SymAlg}(\mathcal{K}, \mathcal{N}, \Pi_{\mathcal{G}}).$$

is a DL-Lite$^{pr}_{\mathcal{R}}$ representation of $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^s_{\subseteq}$. Moreover, $\mathcal{K}'$ computable in time polynomial in $|\mathcal{K} \cup \mathcal{N}|$.

Proof. Since for *DL-Lite$^{pr}_{\mathcal{R}}$* $\mathcal{G}^s_{\subseteq} \equiv_{sem} \mathcal{G}^s_{\#}$ (Theorem 20), it is enough to show the theorem for $\mathcal{G}^s_{\subseteq}$. Let $\mathcal{M}$ be the result of evolution $\mathcal{K} \diamond \mathcal{N}$ wrt $\mathcal{G}^s_{\subseteq}$ and $\mathcal{M}' = Mod(\mathcal{K}')$. Polynomiality of `SymAlg` with $\Pi_{\mathcal{G}}$ follows from polynomiality of `AlignAlg`, the fact that $|\mathcal{S}_2|$ is worst case quadratic in $|\mathcal{N} \cup \mathcal{T}|$, and that the test $\Pi_{\mathcal{G}}(\phi)$ is polynomial in $|\mathcal{K} \cup \mathcal{N}|$.

$\mathcal{M} \subseteq \mathcal{M}'$: Consider a model $\mathcal{J}_0 \in \mathcal{M}$. We show that $\mathcal{J}_0 \in \mathcal{M}'$. By definition of $\mathcal{G}^s_{\subseteq}$ we have $\mathcal{J}_0 \in Mod(\mathcal{T}) \cap Mod(\mathcal{N})$ and there exists a model $\mathcal{I}_0 \in Mod(\mathcal{K})$ such that for every pair of models $\mathcal{J}_1 \in Mod(\mathcal{T}) \cap Mod(\mathcal{N})$ and $\mathcal{I}_1 \in Mod(\mathcal{K})$ Equation 7 *does not* hold. Assume that $\mathcal{J}_0 \notin \mathcal{M}'$. We now exhibit a pair of appropriate $\mathcal{I}_1$ and $\mathcal{J}_1$ that satisfies Equation 7, thus, obtaining a contradiction. Since $\mathcal{J}_0 \notin \mathcal{M}'$ and $\mathcal{J}_0 \models \mathcal{N}$ then, by Line 6 of `SymAlg` (see Algorithm 2), $\mathcal{J}_0 \not\models \mathcal{S}_1$. Hence, there exists a literal $L(c) \in \mathcal{S}_1$ such that $\mathcal{J}_0 \not\models L(c)$. Proposition 14 and the fact that $\mathcal{S}_1 \subseteq \texttt{AlignAlg}(\mathcal{K}, \mathcal{N})$ imply that $L(c)$ is of the form $A(c)$ or $\neg A(c)$. Moreover, from $\mathcal{I}_0 \models \mathcal{K}$ we conclude that $\mathcal{I}_0 \models \mathcal{S}_1$ and consequently $\mathcal{I}_0 \models L(c)$. Therefore, $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$. Now we construct $\mathcal{I}_1$ and $\mathcal{J}_1$ as in Equations 8 and 9, respectively. The proof that Equation 7 holds for these $\mathcal{I}_1$ and $\mathcal{J}_1$ is similar to the one in Theorem 20. Thus, $\mathcal{J}_0 \in \mathcal{M}'$.

$\mathcal{M}' \subseteq \mathcal{M}$: Let $\mathcal{J}_0 \in \mathcal{M}' = Mod(\mathcal{T}) \cap Mod(\mathcal{A}')$ where $\mathcal{A}' = \texttt{AlignAlg}(\mathcal{K}, \mathcal{N}, \Pi_{\mathcal{G}})$, and assume $\mathcal{J}_0 \notin \mathcal{M}$, that is: *(i)* $\mathcal{J}_0 \notin Mod(\mathcal{T}) \cap Mod(\mathcal{N})$, or *(ii)* for every $\mathcal{I} \models \mathcal{K}$ there is a pair of models $\mathcal{I}' \models \mathcal{K}$ and $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ s.t. $dist^s_{\subseteq}(\mathcal{I}', \mathcal{J}') \subsetneq dist^s_{\subseteq}(\mathcal{I}, \mathcal{J}_0)$. Case (i) is impossible since $\mathcal{N} \subseteq \mathcal{A}'$. If Case (ii) holds, then consider a model $\mathcal{I}_0$ as in Equation 5. By Lemma 13 we have $\mathcal{I}_0 \models \mathcal{K}$. By our assumption, $dist^s_{\subseteq}(\mathcal{I}', \mathcal{J}') \subsetneq dist^s_{\subseteq}(\mathcal{I}_0, \mathcal{J}_0)$ holds for some $\mathcal{I}'$ and $\mathcal{J}'$. Due to Proposition 14, $\mathcal{I}_0$ and $\mathcal{J}_0$ coincide on how they interpret roles. Thus, there is a concept A such that $A^{\mathcal{I}'} = A^{\mathcal{J}'}$ while $A^{\mathcal{I}_0} \neq A^{\mathcal{J}_0}$, and consequently there is an atom $A(c) \in \mathcal{I}_0 \ominus \mathcal{J}_0$. We have two cases:

(i) $A(c) \in \mathcal{I}_0 \setminus \mathcal{J}_0$: From $A(c) \in \mathcal{I}_0$ we conclude that $A(c) \in fcl_{\mathcal{T}}(\mathcal{A})$. From $A(c) \notin \mathcal{J}_0$ and $\mathcal{J}_0 \models \mathcal{K}'$, we conclude that $A(c) \notin \mathcal{S}_1$ (see Line 6 in Algorithm 2). From the two statements $A(c) \in fcl_{\mathcal{T}}(\mathcal{A})$ and $A(c) \notin \mathcal{S}_1$, we imply that for some b (which can also be c) one of the two cases holds: $\neg A(b) \in fcl_{\mathcal{T}}(\mathcal{N})$ and $A(b) \in fcl_{\mathcal{T}}(\mathcal{A})$, or $A(b) \in fcl_{\mathcal{T}}(\mathcal{N})$ and $\neg A(b) \in fcl_{\mathcal{T}}(\mathcal{A})$. Either of these cases together with $\mathcal{J}' \models fcl_{\mathcal{T}}(\mathcal{N})$ and $\mathcal{I}' \models fcl_{\mathcal{T}}(\mathcal{A})$ imply $A^{\mathcal{I}'} \neq A^{\mathcal{J}'}$ and yield a contradiction with $A^{\mathcal{I}'} = A^{\mathcal{J}'}$.

(ii) $A(c) \in \mathcal{J}_0 \setminus \mathcal{I}_0$: In this case $A(c) \in \mathcal{J}_0^{\mathcal{A}}$ (see Equation 5) which means that $\{A(c)\} \cup fcl_{\mathcal{T}}(\mathcal{A}) \models_{\mathcal{T}} \bot$ and consequently $\neg A(c) \in fcl_{\mathcal{T}}(\mathcal{A})$. From $\mathcal{J}_0 \models A(c)$ we conclude that $\neg A(c) \notin \mathcal{S}_1$. Finally, from the two statements $\neg A(c) \in fcl_{\mathcal{T}}(\mathcal{A})$ and $\neg A(c) \notin \mathcal{S}_1$ we conclude that $A^{\mathcal{I}'} \neq A^{\mathcal{J}'}$ using the same argument as for $A(c) \in fcl_{\mathcal{T}}(\mathcal{A})$ and $A(c) \notin \mathcal{S}_1$ of Case (i) above.

Thus, $\mathcal{J}_0 \in \mathcal{M}$, which concludes the proof. □

Example 23. Continuing with Example 10, $\texttt{SymAlg}(\mathcal{T}_1^{ex} \cup \mathcal{A}^{ex}, \mathcal{N}_2^{ex}, \Pi_{\mathcal{G}})$ is equal to

$$\{Husb(john),\ Wife(mary),\ Wife(chloe),\ HasHusb(mary, john), \neg Card(john)\} \cup \mathcal{N}_2^{ex}.$$

That is, $\mathcal{K}' = \mathcal{T}_1^{ex} \cup \mathcal{A}^{ex} \diamond \mathcal{N}_2^{ex}$ under $\mathcal{G}_\beta^s$ is equal to $Mod(\mathcal{T}_1^{ex} \cup \texttt{SymAlg}(\mathcal{T}_1^{ex} \cup \mathcal{A}^{ex}, \mathcal{N}_2^{ex}, \Pi_{\mathcal{G}}))$, where $\beta \in \{\subseteq, \#\}$. A closer look at $\mathcal{K}'$ reveals a very counterintuitive behavior of evolution under $\mathcal{G}_\beta^s$: as soon as we declare that a specific object is no longer in a concept, say A, (by asserting that it is in the complement to A, e.g., when we declared that *pedro* is no longer in $Priest$ by asserting that it is in $Husb$), the old information about *all* the other objects in A should be erased (all old members of $Priest$ should be erased). ■

Capturing Local Semantics. Observe that $\mathcal{L}_\subseteq^s$ and $\mathcal{L}_\#^s$ are not expressible in *DL-Lite*$_\mathcal{R}^{pr}$ because they require disjunction, which is not available in *DL-Lite*$_\mathcal{R}$.

Theorem 24. *DL-Lite*$_\mathcal{R}^{pr}$ *is not closed under* $\mathcal{L}_\subseteq^s$ *and* $\mathcal{L}_\#^s$ *semantics.*

Proof. Since $\mathcal{L}_\subseteq^s$ and $\mathcal{L}_\#^s$ coincide for *DL-Lite*$_\mathcal{R}$ (Theorem 20), it suffices to show the claim for $\mathcal{L}_\subseteq^s$. Consider the KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, where: $\mathcal{T} = \{A \sqsubseteq B\}$, $\mathcal{A} = \{B(c)\}$, and $\mathcal{N} = \{B(d)\}$. We now show that every $\mathcal{J} \models \mathcal{K} \diamond \mathcal{N}$ satisfies $A(d) \to B(c)$ and there is a model in $\mathcal{K} \diamond \mathcal{N}$ not satisfying $B(c)$. Due to Lemma 1 in [8] this will give inexpressibility of $\mathcal{M}$ in *DL-Lite*, and hence in *DL-Lite*$_\mathcal{R}^{pr}$. Clearly, $\mathcal{J} = \{B(d)\} \in \mathcal{K} \diamond \mathcal{N}$ and $\mathcal{J} \not\models B(c)$.

Assume there is a model $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ such that $\mathcal{J} \not\models A(d) \to B(c)$, i.e., $A(d) \in \mathcal{J}$ but $B(c) \notin \mathcal{J}$. By definition of $\mathcal{L}_\subseteq^s$, there is $\mathcal{I} \models \mathcal{K}$ such that for every $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ it *does not* hold that $dist_\subseteq^s(\mathcal{I}, \mathcal{J}') \subsetneq dist_\subseteq^s(\mathcal{I}, \mathcal{J})$. Since $\mathcal{I} \models \mathcal{K}$, $B(c) \in \mathcal{I}$, thus $B \in dist_\subseteq^s(\mathcal{I}, \mathcal{J})$. Does $A(d)$ belong to $\mathcal{I}$? If $A(d) \in \mathcal{I}$, then also $B(d) \in \mathcal{I}$, thus $\mathcal{I} \models \mathcal{T} \cup \mathcal{N}$ and by taking $\mathcal{J}' = \mathcal{I}$ one obtains $dist_\subseteq^s(\mathcal{I}, \mathcal{J}') \subsetneq dist_\subseteq^s(\mathcal{I}, \mathcal{J})$ which yields a contradiction. If $A(d) \notin \mathcal{I}$, then $\{A, B\} \in dist_\subseteq^s(\mathcal{I}, \mathcal{J})$. Consider $\mathcal{J}' = \mathcal{I} \cup \{B(d)\}$. Clearly, $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$. Now if $B(d) \in \mathcal{I}$ then $dist_\subseteq^s(\mathcal{I}, \mathcal{J}') = \emptyset$, otherwise $dist_\subseteq^s(\mathcal{I}, \mathcal{J}') = \{B\}$. In either way $dist_\subseteq^s(\mathcal{I}, \mathcal{J}') \subsetneq dist_\subseteq^s(\mathcal{I}, \mathcal{J})$, which yields a contradiction. Thus, every $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ satisfies $A(d) \to B(c)$. □ □

We now show that using $\texttt{SymAlg}$ with the following $\Pi_\mathcal{L}$:

$$\Pi_\mathcal{L}(\phi) \text{ is true if and only if } \phi \notin \mathcal{S}_1.$$

one can compute minimal sound approximations under local symbol-based MBAs. Intuitively $\Pi_\mathcal{L}$ checks whether the ABox $\mathcal{A}$ $\mathcal{T}$-entails assertions $A(c)$ from $fcl_\mathcal{T}(\mathcal{N})$, and if it does not, then the algorithm deletes all the assertions over the concept A from $fcl_\mathcal{T}(\mathcal{A})$. Note that $\Pi_\mathcal{L}$ is weaker than $\Pi_\mathcal{G}$, since it is easier to get changes in the interpretation of A by choosing a model of $\mathcal{K}$ that does not include $A(c)$.

Theorem 25. *Let* $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ *and* $\mathcal{N}$ *be an evolution setting, and* $\mathcal{T}$ *in DL-Lite*$_\mathcal{R}^{pr}$. *Then*

$$\mathcal{K}'' = \mathcal{T} \cup \texttt{SymAlg}(\mathcal{K}, \mathcal{N}, \Pi_\mathcal{L})$$

is a DL-Lite$_\mathcal{R}^{pr}$ *minimal sound approximation of* $\mathcal{K} \diamond \mathcal{N}$ *under both* $\mathcal{L}_\subseteq^s$ *and* $\mathcal{L}_\#^s$. *Moreover,* $\mathcal{K}''$ *is computable in time polynomial in* $|\mathcal{K} \cup \mathcal{N}|$.

Proof. Since for *DL-Lite*$_\mathcal{R}$ $\mathcal{L}_\subseteq^s$ and $\mathcal{L}_\#^s$ coincide (Theorem 20), it suffices to show the claim for $\mathcal{L}_\subseteq^s$. Polynomiality of $\texttt{SymAlg}$ with $\Pi_\mathcal{L}$ can be shown analogously to polynomiality of $\texttt{SymAlg}$ with $\Pi_\mathcal{G}$ (see Theorem 22). The fact that $\mathcal{K}''$ is a sound approximation of $\mathcal{K} \diamond \mathcal{N}$, i.e., $\mathcal{K} \diamond \mathcal{N} \subseteq Mod(\mathcal{K}'')$, can also be shown analogously to soundness in Theorem 22.

Suppose that $\mathcal{K}''$ is not a minimal sound approximation, i.e., we may add an assertion $A(c)$ to A'', where $A(c)$ is such that $\mathcal{K}'' \not\models A(c)$. That is, $\mathcal{K}_1'' = \mathcal{T} \cup \mathcal{A}'' \cup \{A(c)\}$ is another sound approximation. Consider a canonical model $\mathcal{J}''$ of $\mathcal{K}''$. Using a similar argument as in the proof of completeness in Theorem 22, one can show that $\mathcal{J}'' \in \mathcal{K} \diamond \mathcal{N}$. Clearly, $A(c) \notin \mathcal{J}''$, thus $\mathcal{J}'' \not\models \mathcal{K}_1''$ which contradicts soundness of the approximation $\mathcal{K}_1''$. □

We illustrate the behavior of symbol-based MBAs on the following example.

Example 26. Continuing with Example 10, $\texttt{SymAlg}(\mathcal{T}_1^{ex} \cup \mathcal{A}^{ex}, \mathcal{N}_2^{ex}, \Pi_{\mathcal{L}})$ is equal to

$$\{Husb(john), HasHusb(mary, john), \neg Card(john)\} \cup \mathcal{N}_2^{ex}.$$

That is, $\mathcal{K}' = \mathcal{T}_1^{ex} \cup \mathcal{A}^{ex} \diamond \mathcal{N}_2^{ex}$ under $\mathcal{L}_\beta^s$ is equal to $Mod(\mathcal{T}, \texttt{SymAlg}(\mathcal{K}, \mathcal{N}, \Pi_{\mathcal{L}}))$, where $\beta \in \{\subseteq, \#\}$. Observe that $\mathcal{K}'$ in this case is even less intuitive than the one for global semantics $\mathcal{G}_\beta^s$ considered in Example 23: this type of evolution erases all the old ABox information about a concept, say B, (e.g., such a B is $Wife$ in our case) as soon as we just add any new object in B that does not even conflict with anything in the old ABox (in our case we added $Wife(tanya)$ and had to erase the other two wives $Wife(mary)$ and $Wife(chloe)$ from the old knowledge). ■

To sum up on *DL-Lite*$_{\mathcal{R}}^{pr}$: atom-based approaches (which all coincide) can be captured using a polynomial-time algorithm based on the alignment algorithm. Moreover, the evolution results produced under these MBAs are intuitive and expected, e.g., see Examples 23 and 26, while symbol-based approaches produce quite unexpected and counterintuitive results (these semantics delete too much data). Furthermore, two out of four of the latter approaches cannot be captured in *DL-Lite*$_{\mathcal{R}}^{pr}$. Based on these results we conclude that using atom-based approaches for applications seem to be more practical. In Figure 2 we framed in a dashed rectangle the six out of eight MBAs under which *DL-Lite*$_{\mathcal{R}}^{pr}$ is closed.

5. $\mathcal{L}_\subseteq^a$ Evolution of *DL-Lite*$_{\mathcal{R}}$ KBs

In the previous section we showed that atom-based MBAs behave well for *DL-Lite*$_{\mathcal{R}}^{pr}$ evolution settings, while symbol-based ones do not. This suggests to investigate atom-based MBAs for the entire *DL-Lite*$_{\mathcal{R}}$. Here we focus on one of these four semantics, namely $\mathcal{L}_\subseteq^a$. The remaining three atom-based MBAs are subject of future work.

As a further motivation for the study of $\mathcal{L}_\subseteq^a$, note that $\mathcal{L}_\subseteq^a$ is essentially the same as so-called *Winslett's semantics* [23] (WS), that was widely studied in the literature [18, 10]. Liu, Lutz, Milicic, and Wolter studied WS for expressive DLs [18], and KBs with an empty TBox. Most of the DLs they considered are not closed under WS. Poggi, Lembo, De Giacomo, Lenzerini, and Rosati studied WS in the same setting as the one adopted in this paper. They called it instance level update for *DL-Lite* [10] and proposed an algorithm to compute the result of updates However, the algorithm turned out to have technical issues, and it was shown that it is neither sound nor complete [8]. Note that extension of this algorithm that approximates ABox updates in fragments of *DL-Lite* [10], inherits these technical issues. Actually, such an ABox update algorithm cannot exist since it was shown that *DL-Lite* is not closed under ABox evolution wrt $\mathcal{L}_\subseteq^a$ [9].

The remaining part of the section is organized as follows. In Section 5.1 we explain *why* *DL-Lite*$_{\mathcal{R}}$ is not closed under $\mathcal{L}_\subseteq^a$ and show which combination of *DL-Lite*$_{\mathcal{R}}$ formulas is responsible for inexpressibility. In Section 5.2 we introduce so-called prototypes that give a characterization of $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}_\subseteq^a$ and are further used to approximate $\mathcal{L}_\subseteq^a$ evolution. In Section 5.3 we we show how to approximate $\mathcal{L}_\subseteq^a$ evolution and give practical consideration.

5.1. Understanding Inexpressibility of Evolution in DL-Lite$_{\mathcal{R}}$

Using the following example we illustrate why $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ is not expressible in *DL-Lite$_{\mathcal{R}}$*.

Example 27. Consider a *DL-Lite$_{\mathcal{R}}$* KB $\mathcal{K}_2^{ex} = \mathcal{T}_2^{ex} \cup \mathcal{A}_2^{ex}$, $\mathcal{N}_1^{ex}$ as the new information, and $\mathcal{I} \models \mathcal{K}_2^{ex}$:

$$\begin{aligned}
\mathcal{T}_2^{ex} &= \{\, \mathit{Wife} \sqsubseteq \exists \mathit{HasHusb}, \exists \mathit{HasHusb}^- \sqsubseteq \neg \mathit{Priest}\};\\
\mathcal{A}_2^{ex} &= \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan}), \exists \mathit{HasHusb}^-(\mathit{john})\};\\
\mathcal{N}_1^{ex} &= \{\mathit{Priest}(\mathit{john})\};
\end{aligned}$$

$$\mathcal{I}: \quad \mathit{Wife}^{\mathcal{I}} = \{\_\mathit{girl}\}, \quad \mathit{Priest}^{\mathcal{I}} = \{\mathit{pedro}, \mathit{ivan}\}, \quad \mathit{HasHusb}^{\mathcal{I}} = \{(\_\mathit{girl}, \mathit{john})\},$$

where $\_\mathit{girl} \in \Delta \setminus \mathit{adom}(\mathcal{K}_2^{ex})$ is an element of the domain. The following models belong to $\mathcal{I} \diamond \mathcal{N}_1^{ex}$:

$$\begin{array}{llll}
\mathcal{J}_0: & \mathit{Wife}^{\mathcal{J}_0} = \emptyset, & \mathit{Priest}^{\mathcal{J}_0} = \{\mathit{john}, \mathit{pedro}, \mathit{ivan}\}, & \mathit{HasHusb}^{\mathcal{J}_0} = \emptyset,\\
\mathcal{J}_1: & \mathit{Wife}^{\mathcal{J}_1} = \{_\mathit{girl}\}, & \mathit{Priest}^{\mathcal{J}_1} = \{\mathit{john}, \mathit{ivan}\}, & \mathit{HasHusb}^{\mathcal{J}_1} = \{(_\mathit{girl}, \mathit{pedro})\},\\
\mathcal{J}_2: & \mathit{Wife}^{\mathcal{J}_2} = \{_\mathit{girl}\}, & \mathit{Priest}^{\mathcal{J}_2} = \{\mathit{john}, \mathit{pedro}\}, & \mathit{HasHusb}^{\mathcal{J}_2} = \{(_\mathit{girl}, \mathit{ivan})\},\\
\mathcal{J}_3: & \mathit{Wife}^{\mathcal{J}_3} = \{_\mathit{girl}\}, & \mathit{Priest}^{\mathcal{J}_3} = \{\mathit{john}, \mathit{pedro}, \mathit{ivan}\}, & \mathit{HasHusb}^{\mathcal{J}_3} = \{(_\mathit{girl}, _\mathit{guy})\},
\end{array}$$

where $\_\mathit{guy} \in \Delta \setminus \mathit{adom}(\mathcal{K}_2^{ex}) \setminus \{\_\mathit{girl}\}$ is an element of the domain.

Indeed, all $\mathcal{J}_i$'s satisfy $\mathcal{N}_1^{ex}$ and $\mathcal{K}_2^{ex}$. To see that they are in $\mathcal{I} \diamond \mathcal{N}_1^{ex}$ observe that every model $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}_1^{ex}$ can be obtained from $\mathcal{I}$ by making modifications that guarantee that $\mathcal{J} \models \mathcal{N}_1^{ex} \cup \mathcal{K}_2^{ex}$ and that the distance between $\mathcal{I}$ and $\mathcal{J}$ is minimal. What are these modifications? Clearly, $\mathit{Priest}(\mathit{john})$ should hold in $\mathcal{J}$. Moreover, no priest can be in the $\mathit{HasHusb}$ relation since $(\mathit{Priest} \sqsubseteq \neg \exists \mathit{HasHusb}^-) \in \mathcal{K}_2^{ex}$. Hence, john cannot be in the $\mathit{HasHusb}$ relation with $\_\mathit{girl}$ after evolution, and the first necessary modification in $\mathcal{I}$ is to drop the atom $\mathit{HasHusb}(\_\mathit{girl}, \mathit{john})$ and to add the atom $\mathit{Priest}(\mathit{john})$:

$$\mathcal{I}' = (\mathcal{I} \setminus \{\mathit{HasHusb}(\_\mathit{girl}, \mathit{john})\}) \cup \{\mathit{Priest}(\mathit{john})\}.$$

Observe that this modification is not enough, i.e., $\mathcal{I}' \notin \mathcal{I} \diamond \mathcal{N}_1^{ex}$, since $\mathcal{I}'$ does not satisfy the TBox, namely, the assertion $\mathit{Wife} \sqsubseteq \exists \mathit{HasHusb}$. Indeed, $\_\mathit{girl}$ is still a wife in $\mathcal{I}'$, while there is no husband for her, that is, no atom of the form $\mathit{HasHusb}(\_\mathit{girl}, x)$ for some x is in $\mathcal{I}'$. We can solve this problem by either dropping $\mathit{Wife}(\_\mathit{girl})$ from $\mathcal{I}'$ or by finding her a husband, that is, adding $\mathit{HasHusb}(\_\mathit{girl}, x)$ to $\mathcal{I}'$. The model $\mathcal{J}_0$ corresponds to the former option and the other three $\mathcal{J}_i$'s correspond to the latter one. That is:

$$\mathcal{J}_0 = \mathcal{I}' \setminus \{\, \mathit{Wife}(\_\mathit{girl})\}. \tag{10}$$

Regarding the other option, who should be the husband of $\_\mathit{girl}$ in $\mathcal{J}$? There are two possibilities in general: either one of the two priests (pedro or ivan), or some other $\_\mathit{guy}$. Clearly, if a priest, say pedro, is a husband of $\_\mathit{girl}$ in $\mathcal{J}$, then he should quit the priesthood due to the TBox assertion $\mathit{Priest} \sqsubseteq \neg \mathit{HasHusb}^-$, i.e., $\mathit{Priest}(\mathit{pedro})$ should not be in $\mathcal{J}$. Thus, further modifications corresponding to the three possibilities are

$$\begin{aligned}
\mathcal{J}_1 &= (\mathcal{J}_0 \setminus \{\mathit{Priest}(\mathit{pedro})\}) \cup (\{\mathit{HasHusb}(_\mathit{girl}, \mathit{pedro})\} \cup \{\, \mathit{Wife}(_\mathit{girl})\}), && (11)\\
\mathcal{J}_2 &= (\mathcal{J}_0 \setminus \{\mathit{Priest}(\mathit{ivan})\}) \cup (\{\mathit{HasHusb}(_\mathit{girl}, \mathit{ivan})\} \cup \{\, \mathit{Wife}(_\mathit{girl})\}). && (12)\\
\mathcal{J}_3 &= (\mathcal{J}_0 \setminus \emptyset) \cup (\{\mathit{HasHusb}(_\mathit{girl}, _\mathit{guy})\} \cup \{\, \mathit{Wife}(_\mathit{girl})\}).
\end{aligned}$$

Note that we wrote the three formulas above in a specific way: first we subtract atoms about Priest from $\mathcal{J}_0$ (whenever it is needed) and then we add $\mathit{HasHusb}$ and Wife-atoms that are required to comply with the TBox $\mathcal{K}_2^{ex}$. This is done in order to be coherent with the general procedure of `BP` that we will use to compute $\mathcal{K} \diamond \mathcal{N}$ (see Section 5.3 for details). ■

Lack of Canonical Models. Recall that for every *DL-Lite*$_\mathcal{R}$ KB $\mathcal{K}$, the set *Mod*($\mathcal{K}$) has a canonical model. At the same time, continuing with Example 27, one can verify that any model $\mathcal{J}_{can}$ that can be homomorphically embedded into the four $\mathcal{J}_i$s is such that $\mathit{Wife}^{\mathcal{J}_{can}} = \mathit{HasHusb}^{\mathcal{J}_{can}} = \emptyset$, and $\mathit{pedro} \notin \mathit{Priest}^{\mathcal{J}_{can}}$ and $\mathit{ivan} \notin \mathit{Priest}^{\mathcal{J}_{can}}$. It is easy to check that any such $\mathcal{J}_{can}$ is *not* in $\mathcal{K}_1 \diamond \mathcal{N}_1$. Thus, there is no canonical model in $\mathcal{K}_2^{ex} \diamond \mathcal{N}_1^{ex}$ and this set is not expressible in *DL-Lite*$_\mathcal{R}$. This gives us a first reason why *DL-Lite*$_\mathcal{R}$ is not closed under $\mathcal{L}^a_\subseteq$-evolution.

Local Functionality. Another problem with models $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$ is that they entail a special kind of functionality constraints on roles. Let R be a role and c a constant, then we call *local functionality of R on c* the formula

$$\mathsf{func}(R, c) \doteq \forall x \forall y. \, (R(x, c) \land R(x, y) \to y = c) \, .$$

Example 28. Continuing with Example 27, one can see that $\mathcal{K}_2^{ex} \diamond \mathcal{N}_1^{ex}$ entails local functionality of $\mathit{HasHusb}$ on both priests pedro and ivan, for example:

$$\mathsf{func}(\mathit{HasHusb}, \mathit{pedro}) = \forall x \forall y. [(\mathit{HasHusb}(x, \mathit{pedro}) \land \mathit{HasHusb}(x, y) \to (y = \mathit{pedro})) \, .$$

That is, if in $\mathcal{J} \in \mathcal{K}_2^{ex} \diamond \mathcal{N}_1^{ex}$ either pedro or ivan is a husband of $\_\mathit{girl}$, then she cannot be married to anyone else. For example, the following model $\mathcal{J}'$, which violates the local functionality, is *not* in $\mathcal{K}_2^{ex} \diamond \mathcal{N}_1^{ex}$ since it is not minimally distant from $\mathcal{I}$ (or any other model of $\mathcal{K}_2^{ex}$):

$$\mathit{Wife}^{\mathcal{J}'} = \{\_\mathit{girl}\}, \quad \mathit{Priest}^{\mathcal{J}'} = \{\mathit{john}, \mathit{ivan}\}, \quad \mathit{HasHusb}^{\mathcal{J}'} = \{(\_\mathit{girl}, \mathit{pedro}), (\_\mathit{girl}, \_\mathit{guy})\}.$$

At the same time if $\_\mathit{girl}$ has a husband in $\mathcal{J}$ who is neither pedro nor ivan she can be married to more people. For example, the following model $\mathcal{J}''$ is in $\mathcal{K}_2^{ex} \diamond \mathcal{N}_1^{ex}$:

$$\mathit{Wife}^{\mathcal{J}''} = \{\_\mathit{girl}\}, \; \mathit{Priest}^{\mathcal{J}''} = \{\mathit{john}, \mathit{pedro}, \mathit{ivan}\}, \; \mathit{HasHusb}^{\mathcal{J}''} = \{(\_\mathit{girl}, \_\mathit{guy}_1), (\_\mathit{girl}, \_\mathit{guy}_2)\}. \blacksquare$$

The following proposition shows that local functionality is not expressible in *DL-Lite*$_\mathcal{R}$

Proposition 29. *Let R be a role and c a constant. Then $\mathcal{K} \not\models \mathsf{func}(R, c)$ for every DL-Lite$_\mathcal{R}$ KB $\mathcal{K}$ such that $\mathcal{K} \not\models \neg\exists R^-(c)$*[2].

As a corollary of the proposition above, since the set $\mathcal{K}_2^{ex} \diamond \mathcal{N}_1^{ex}$ entails local functionality, it is not expressible in *DL-Lite*$_\mathcal{R}$. This gives us a second argument why *DL-Lite*$_\mathcal{R}$ is not closed under $\mathcal{L}^a_\subseteq$-evolution.

Dually-Affected Roles. Both lack of canonical models and local functionality for $\mathcal{T} \cup \mathcal{A} \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$ observed above come from the fact that there exist a role R, a concept A, and a constant c such that:

$$\mathcal{T} \models A \sqsubseteq \exists R, \qquad \mathcal{T} \cup \mathcal{A} \not\models \neg\exists R^-(c), \quad \text{and} \quad \mathcal{T} \cup \mathcal{N} \models \neg\exists R^-(c). \tag{13}$$

Due to the first and second entailments there is a model $\mathcal{I} \models \mathcal{T} \cup \mathcal{A}$ such that $\mathcal{I} \models A(b)$ and $\mathcal{I} \models R(b, c)$ for some $b \in \Delta$. Due to the third entailment, in the evolution result $\mathcal{I} \diamond \mathcal{N}$ this R-arc from b cannot go to c anymore, and we have to do a case analysis on where it can go. This case analysis results in a set of models that is non-axiomatizable in *DL-Lite*$_\mathcal{R}$.

[2] Another condition would clearly be coherency of R, but we do not mention it in the proposition since in this paper we consider coherent KBs only.

Example 30. In Example 27 clearly $\mathcal{T}_2^{ex} \models Wife \sqsubseteq HasHusb$, $\mathcal{T}_2^{ex} \cup \mathcal{A}_2^{ex} \not\models \exists HasHusb^-(john)$, and $\mathcal{T}_2^{ex} \cup \mathcal{N}_1^{ex} \models \neg\exists HasHusb^-(john)$. Besides, $Wife(\_girl)$ is in $\mathcal{I}$. Thus, $john$ cannot be a husband of $\_girl$ in the models of $\mathcal{I} \diamond \mathcal{N}_2^{ex}$ and we have to do the case analysis on who is her husband: ether nobody, or $pedro$, or $ivan$, or some other $\_guy$. ■

Since we assumed that $\mathcal{N}$ contains only positive ABox assertions, the only way how Equation 13 can hold is if R is what we call dually-affected and triggered.

Definition 31 (Dual-Affection and Triggering)**.** Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ be a *DL-Lite*$_\mathcal{R}$ KB. Then a role R is *dually-affected* in $\mathcal{T}$ via an atomic concept A' if for some atomic concept A it holds that $\mathcal{T} \models A \sqsubseteq \exists R$ and $\mathcal{T} \models \exists R^- \sqsubseteq \neg A'$. A role R that is dually-affected via A' is *triggered* by $\mathcal{N}$ if $\mathcal{A} \not\models_\mathcal{T} \neg\exists R^-(b)$ and $\mathcal{N} \models_\mathcal{T} A'(b)$, for some constant $b \in adom(\mathcal{K} \cup \mathcal{N})$. ■

Example 32. In Example 27, the role $HasHusb$ is dually affected in $\mathcal{T}_2^{ex}$ via $A' = Priest$, with $A = Wife$. Also it is triggered with $\{Priest(john)\}$. ■

The following theorem shows that if there is a dually affected role, we can always find $\mathcal{A}$ and $\mathcal{N}$ to trigger it and thus, to guarantee that $\mathcal{T} \cup \mathcal{A} \diamond \mathcal{N}$ is inexpressible in *DL-Lite*$_\mathcal{R}$.

Theorem 33. *Let $\mathcal{T}$ be a DL-Lite$_\mathcal{R}$ TBox and R a role dually affected in $\mathcal{T}$. Then there are ABoxes $\mathcal{A}$ and $\mathcal{N}$ such that $\mathcal{T} \cup \mathcal{A} \diamond \mathcal{N}$ under $\mathcal{L}_\subseteq^a$ is inexpressible in DL-Lite$_\mathcal{R}$.*

Proof. By definition, there are concepts A and C such that $\mathcal{T} \models A \sqsubseteq \exists R$ and $\mathcal{T} \models \exists R^- \sqsubseteq \neg C$. Now it is enough to take $\mathcal{A}$ and $\mathcal{N}$ analogous to, respectively, $\mathcal{A}_2^{ex}$ and $\mathcal{N}_1^{ex}$ from Example 27. Then $\mathcal{T} \cup \mathcal{A} \diamond \mathcal{N}$ is non-axiomatizable in *DL-Lite*$_\mathcal{R}$ since it has no canonical model and entails local functionality which by Proposition 29 prevents expressibility in *DL-Lite*$_\mathcal{R}$. □

5.2. Prototypes

As we discussed above, the set of models $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}_\subseteq^a$ may not have a canonical model. A closer look at $\mathcal{K} \diamond \mathcal{N}$ gives a surprising result: this set can be divided (but in general not partitioned) into a finite number of subsets $\mathcal{S}_0, \ldots, \mathcal{S}_n$, that is, $\mathcal{S}_i \subseteq \mathcal{K} \diamond \mathcal{N}$ for each $i \in \{1, \ldots, n\}$ and $\cup_{i=1}^n \mathcal{S}_i = \mathcal{K} \diamond \mathcal{N}$, where each $\mathcal{S}_i$ includes its canonical model $\mathcal{J}_i$. Each of this $\mathcal{J}_i$ is a minimal element in $\mathcal{K} \diamond \mathcal{N}$ wrt homomorphisms. Moreover, n is worst-case exponential in $|\mathcal{K} \cup \mathcal{N}|$. We now discuss this aspect in detail.

Definition 34 (Prototypal Sets)**.** Let $\mathcal{M}$ be a set of models. A *prototypal set for $\mathcal{M}$* is a minimal finite subset $\mathfrak{J} = \{\mathcal{J}_0, \ldots, \mathcal{J}_n\}$ of $\mathcal{M}$ satisfying the following property: for every $\mathcal{J} \in \mathcal{M}$ there exists $\mathcal{J}_i \in \mathfrak{J}$ that homomorphically embeddable into $\mathcal{J}$. ■

Each $\mathcal{J}_i$ in $\mathfrak{J}$ is called a *prototype for $\mathcal{M}$*. The notion of prototypes generalizes the notion of canonical model: for example, if $\mathcal{K}$ is a *DL-Lite*$_\mathcal{R}$ KB, then a prototypal set $\mathfrak{J}$ for *Mod*$(\mathcal{K})$ is equal to $\{\mathcal{I}^{can}\}$. Clearly, an arbitrary set of models may not have a prototypal set. We will say that $\mathfrak{J}$ is a *prototypal set for $\mathcal{K}$ and $\mathcal{N}$* if it is for $\mathcal{K} \diamond \mathcal{N}$.

Example 35. Continuing with Example 27, one can check that the prototypal set for $\mathcal{K}_2^{ex}$ and $\mathcal{N}_1^{ex}$ is $\{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_4\}$, where $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ are as in Example 27, and $\mathcal{J}_4$ is as follows: $Wife^{\mathcal{J}_4} = \{\_girl_1, \_girl_2\}$, $Priest^{\mathcal{J}_4} = \{john\}$, and $HasHusb^{\mathcal{J}_4} = \{(\_girl_1, pedro), (\_girl_2, ivan)\}$. Note that $\mathcal{J}_3$ is not a prototype for $\mathcal{K}_2^{ex}$ and $\mathcal{N}_1^{ex}$ since $\mathcal{J}_0 \subsetneq \mathcal{J}_3$, and hence $\mathcal{J}_0$ is homomorphically embeddable in $\mathcal{J}_3$. Also the prototypes $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ were obtained by manipulations with the model $\mathcal{I}$ from Example 27, while $\mathcal{J}_4$ can be obtained from the interpretation $\mathcal{I}' \models \mathcal{K}_2^{ex}$, where $Wife^{\mathcal{I}'} = \{\_girl_1, \_girl_2\}$, $Priest^{\mathcal{I}'} = \{pedro, ivan\}$, and $HasHusb^{\mathcal{I}'} = \{(\_girl_1, john), (\_girl_2, john)\}$. Finally observe that evolution of $\mathcal{I}'$ with $\mathcal{N}_1^{ex}$ yields all four prototypes. ■

The following proposition shows that for *every* evolution setting $\mathcal{K}$ and $\mathcal{N}$ there is a prototypal set.

Proposition 36. *Let $\mathcal{K}$ and $\mathcal{N}$ be an evolution setting. Then there exists a prototypal set for $\mathcal{K}$ and $\mathcal{N}$ that is of size exponential in $|\mathcal{K} \cup \mathcal{N}|$.*

We now introduce a procedure BP (where BP stands for *Build Prototypes*) that takes $\mathcal{K}$ and $\mathcal{N}$ as input and returns the prototypal set for $\mathcal{K}$ and $\mathcal{N}$. The proof of Proposition 36 will follow from the correctness of this procedure (see Theorem 39). Before proceeding to BP we shall introduce several notions and notations that the procedure is based upon.

For the ease of exposition of BP we consider a restricted form of evolution settings. An evolution setting $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ is *simple*, if *(i)* for every role R there is *no* atomic concept A such that $\mathcal{T} \models \exists R \sqsubseteq A$, *(ii)* for every two different roles R and R', neither $\mathcal{T} \models \exists R \sqsubseteq \exists R'$ nor $\mathcal{T} \models \exists R \sqsubseteq \neg\exists R'$ holds, and *(iii)* If $\mathcal{N} \models_{\mathcal{T}} \exists R(a)$, then $\mathcal{N} \models_{\mathcal{T}} R(a, b)$ for some constant b. *(iv)* if for an atomic concept D there is B and R such that $B \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq \neg D$ are in $cl(\mathcal{T})$, then for every R' it holds that $D \sqsubseteq \exists R' \notin cl(\mathcal{T})$. Note that Case (ii) still allows some interaction between two roles in TBoxes, e.g., role projections may contain the same concept. These three restrictions allow us to analyze evolution that affects roles independently for every role. We will comment later on how the following techniques can be extended to the case of *DL-Lite*$_\mathcal{R}$, where roles can interact arbitrarily.

Components of BP *Procedure.* Let $\mathcal{T}$ be a *DL-Lite*$_\mathcal{R}$ TBox, $\mathcal{N}$ an ABox with only positive assertions and satisfiable with $\mathcal{T}$, and $\mathcal{I}$ a model of $\mathcal{T}$. Then a *$\mathcal{T}$-alignment of $\mathcal{I}$ with $\mathcal{N}$*, denoted $Align_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$ (or just *alignment* when $\mathcal{T}$ is clear), is defined as follows:

$$Align_{\mathcal{T}}(\mathcal{I}, \mathcal{N}) = \mathcal{I} \setminus \bigcup_{g \in \mathcal{I} \text{ s.t. } \{g\} \cup \mathcal{N} \models_{\mathcal{T}} \bot} root_{\mathcal{T}}(g).$$

Note that $Align_{\mathcal{T}}$ in a way extends AlignAlg from ABoxes to models.

Let $TR[\mathcal{K}, \mathcal{N}]$ (where *TR* stands for *triggered roles*), or simply *TR* when $\mathcal{T}$ and $\mathcal{N}$ are clear, be the set of all roles dually-affected in $\mathcal{T}$ that are triggered by $\mathcal{N}$. In Example 27, $TR[\mathcal{K}_2^{ex}, \mathcal{N}_1^{ex}] = \{HasHusb\}$.

Let R be a role occurring in $\mathcal{K}$ that is dually-affected and triggered by $\mathcal{N}$. Then the set of unary atoms $\texttt{DjnAts}[\mathcal{K}, \mathcal{N}](R) \subseteq fcl_{\mathcal{T}}(\mathcal{A})$ (where DjnAts stands for *Disjoint Atoms*) contains $D(c)$ if $\mathcal{T}$ entails that the range of R is disjoint with D, while $\mathcal{N}$ "says" nothing about $D(c)$. Formally:

$$\begin{aligned}\texttt{DjnAts}[\mathcal{K}, \mathcal{N}](R) = \{D(c) \in fcl_{\mathcal{T}}(\mathcal{A}) \mid & R \in TR[\mathcal{K}, \mathcal{N}],\ \{\exists R^-(c), D(c)\} \models_{\mathcal{T}} \bot, \\ & \mathcal{N} \not\models_{\mathcal{T}} D(c), \text{ and } \mathcal{N} \not\models_{\mathcal{T}} \neg D(c)\}.\end{aligned}$$

In Example 27, $\texttt{DjnAts}[\mathcal{K}_2^{ex}, \mathcal{N}_1^{ex}](HasHusb) = \{Priest(pedro), Priest(ivan)\}$. The set of disjoint atoms for the entire KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$, denoted $\texttt{DjnAts}(\mathcal{K}, \mathcal{N})$, or DjnAts when the parameters are clear, is $\bigcup_{R \in TR} \texttt{DjnAts}[\mathcal{K}, \mathcal{N}](R)$.

For a role R, the set $\texttt{ISubCon}[\mathcal{T}](\exists R)$ (where ISubCon stands for *Immediate Sub-Concepts*) is the set of atomic concepts that are subsumed by $\exists R$ and are "immediately" under $\exists R$ in the concept hierarchy generated by $\mathcal{T}$. Formally:

$$\texttt{ISubCon}[\mathcal{T}](\exists R) = \{A \mid \mathcal{T} \models A \sqsubseteq \exists R \text{ and there is no } A' \neq A \text{ s.t. } \mathcal{T} \models A \sqsubseteq A' \text{ and } \mathcal{T} \models A' \sqsubseteq \exists R\}.$$

In Example 27, $\texttt{ISubCon}[\mathcal{T}_2^{ex}](\exists HasHusb) = \{Wife\}$.

We are ready to proceed to the description of the BP procedure. It works similar to the way we described in Equations 10-12 of Example 27, that is, by first constructing one prototype $\mathcal{J}_0$ by aligning $\mathcal{I}^{can}$ of $\mathcal{K}$ with $\mathcal{N}$ (recall that in $\mathcal{J}_0$ of Example 27 *_girl* is not a *Wife* anymore and all the priests of $\mathcal{I}$ remain priests), and then manipulating $\mathcal{J}_0$ in order to get all the other prototypes. We will further refer to such a model $\mathcal{J}_0$ as the *zero prototype*. We start with a procedure BZP for constructing the zero prototype.

$\texttt{BZP}(\mathcal{K}, \mathcal{N})$

1. $\mathcal{J}_0 := \mathit{Align}_{\mathcal{T}}(\mathcal{I}^{can}, \mathcal{N})$, where $\mathcal{I}^{can}$ is the canonical model of $\mathcal{K}$.
2. For each $R \in \mathit{TR}[\mathcal{K}, \mathcal{N}]$ do
   for each $A \in \texttt{ISubCon}[\mathcal{K}](\exists R)$ do
   if $A(x) \in \mathcal{J}_0$ for some $x \in \Delta$, and for every $b \in \mathit{adom}(\mathcal{K} \cup \mathcal{N})$: $\mathcal{J}_0 \not\models_{\mathcal{T}} R(x, b), \mathcal{N} \not\models_{\mathcal{T}} R(x, b)$
   then $\mathcal{J}_0 := \mathcal{J}_0 \setminus (\mathit{root}_{\mathcal{T}}(A(x)) \cup \bigcup_{y \in \Delta \setminus \mathit{adom}(\mathcal{K})} \{R(x, y)\})$,
3. $\mathcal{J}_0 := \mathit{chase}_{\mathcal{T}}(\mathcal{J}_0 \cup \mathcal{N})$,
4. Return $\mathcal{J}_0$.

Figure 4: The procedure $\texttt{BZP}(\mathcal{K}, \mathcal{N})$ for building the zero prototype $\mathcal{J}_0$ for a simple evolution setting $\mathcal{K} = \mathcal{T} \cup \mathcal{A})$ and $\mathcal{N}$

Procedure $\texttt{BZP}$. The procedure $\texttt{BZP}(\mathcal{K}, \mathcal{N})$ (where $\texttt{BZP}$ stands for *Build Zero Prototype*) in Figure 4 constructs the zero prototype $\mathcal{J}_0$ for $\mathcal{K}$ and $\mathcal{N}$. It works as follows. First, it deletes from the canonical model $\mathcal{I}^{can}$ of $\mathcal{K}$ all the atoms that are not $\mathcal{T}$-satisfiable with $\mathcal{N}$ (Step 1). Then, in Step 2 the procedure does the following: from the interpretation $\mathcal{J}_0$ resulting in Step 1 it deletes all atoms of the form $A(a)$ (together with the atoms that $\mathcal{T}$-entail $A(a)$) for which there is no constant b from $\mathit{adom}(\mathcal{K} \cup \mathcal{N})$ such that $\mathcal{J}_0 \models R(a, b)$ or $\mathcal{N} \models_{\mathcal{T}} R(a, b)$. Moreover, it further deletes from $\mathcal{J}_0$ all atoms of the form $R(a, x)$ where $x \in \Delta \setminus \mathit{adom}(\mathcal{K} \cup \mathcal{N})$. Intuitively, Step 2 works as follows: if neither $\mathcal{J}_0$ nor $\mathcal{N}$ entails an atom of the form $R(a, b)$ (i.e., there is no active-domain husband b of a *_girl* provided by $\mathcal{J}_0$ or $\mathcal{N}_1^{ex}$ in Example 27), then the zero prototype should not contain $A(a)$ (i.e., then *_girl* stops to be a wife in Example 27) and also all atoms $R(a, x)$ for some non-active x. Step 3 combines $\mathcal{J}_0$ resulting from Step 2 with $\mathcal{N}$ and chases them in order to obtain a model of $\mathcal{T} \cup \mathcal{N}$. Finally, Step 4 returns $\mathcal{J}_0$.

We illustrate $\texttt{BZP}$ on the following example.

Example 37. In Example 27 the zero prototype obtained by $\texttt{BZP}(\mathcal{K}_2^{ex}, \mathcal{N}_1^{ex})$ is $\mathcal{J}_0$. Indeed, the canonical model of $\mathcal{K}_2^{ex}$ is $\mathcal{I}^{can} = \{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan}), \mathit{HasHusb}(x, \mathit{john})\}$. Step 1 of $\texttt{BZP}(\mathcal{K}_1, \mathcal{N}_1)$ returns $\mathcal{I}^{can} \setminus \{\mathit{HasHusb}(x, \mathit{john})\}$ and Step 2 deletes $\mathit{Wife}(\_girl)$ from $\mathcal{J}_0$. Finally, Step 3 returns the interpretation $\mathit{chase}(\{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan})\} \cup \{\mathit{Priest}(\mathit{john})\})$, which coincides with $\mathcal{J}_0$ of Example 27.

Consider another example: $\mathcal{A} = \{C(a)\}$, $\mathcal{T} = \{C \sqsubseteq A, A \sqsubseteq \exists R, \exists R^- \sqsubseteq \neg B\}$, and $\mathcal{N} = \{B(b)\}$. Then $\mathcal{I}^{can} = \{C(a), A(a), R(a, x)\}$ and Step 1 of $\texttt{BZP}(\mathcal{K}, \mathcal{N})$ returns the model $\mathcal{I}^{can}$. Then Step 2 deletes from $\mathcal{I}^{can}$ the atom $R(a, x)$ and $\mathit{root}_{\mathcal{T}}(A(a)) = \{C(a), A(a)\}$, thus, it returns $\emptyset$. Finally, Step 3 returns $\mathcal{J}_0 = \mathit{chase}_{\mathcal{T}}\{\emptyset \cup \{B(b)\}\} = \{B(b)\}$. ■

Procedure $\texttt{BP}$ *for Building Prototypes.* The procedure $\texttt{BP}(\mathcal{K}, \mathcal{N})$ for constructing $\mathfrak{J}$ (see Figure 5) takes $\mathcal{K}$ and $\mathcal{N}$ as input, constructs the zero prototype $\mathcal{J}_0$ by calling $\texttt{BZP}$ (at Step 1), and based on $\mathcal{J}_0$ builds the other prototypes of $\mathfrak{J}$ (Step 2). Each element in $\mathfrak{J}$ corresponds to a distinct triple consisting of a set $\mathcal{D}$ and two tuples $\mathcal{R}$ (depending on $\mathcal{D}$) and $\mathcal{B}$ (depending on $\mathcal{R}$) that are constructed from $\mathcal{K}$ and $\mathcal{N}$. Thus, $\texttt{BP}$ first chooses a triple $\mathcal{D}$, $\mathcal{R}$, $\mathcal{B}$ that is composed of

(i) a set $\mathcal{D}$ of disjoint atoms from $\texttt{DjnAts}[\mathcal{K}, \mathcal{N}]$ (in Example 27, $\mathcal{D}$ is any subset of the priests from $\mathcal{A}$, that is, of $\{\mathit{Priest}(\mathit{pedro}), \mathit{Priest}(\mathit{ivan})\}$);

(ii) a tuple $\mathcal{R}$ of roles R, one for each $D(c) \in \mathcal{D}$, such that $D(c)$ is a disjoint atom for R, that is, $D(c) \in \texttt{DjnAts}(R)$ (in Example 27, $\mathcal{R} = \langle \mathit{HasHusb} \rangle$ for every possible $\mathcal{D}$ since $D(c)$ can be either $\mathit{Priest}(\mathit{pedro})$ or $\mathit{Priest}(\mathit{ivan})$ and it holds that $\mathit{Priest}(\mathit{pedro}) \in \texttt{DjnAts}(\mathit{HasHusb})$ and $\mathit{Priest}(\mathit{ivan}) \in \texttt{DjnAts}(\mathit{HasHusb})$);

(iii) a tuple $\mathcal{B}$ of immediate subconcepts A of $\exists R$ for each $R \in \mathcal{R}$ (in Example 27, $\mathcal{B} = \langle \mathit{Wife} \rangle$ since Wife is the only immediate subconcept of $\mathit{HasHusb}$).

$\texttt{BP}(\mathcal{K},\mathcal{N})$

1. $\mathcal{J}_0 := \texttt{BZP}(\mathcal{K},\mathcal{N})$
2. For each set $\mathcal{D} = \{D_1(c_1),\ldots,D_k(c_k)\} \subseteq \texttt{DjnAts}[\mathcal{K},\mathcal{N}]$ do
   for each vector $\mathcal{R} = \langle R_1,\ldots,R_k\rangle$, s.t. $R_j \in \mathit{TR}$ and $D_j(c_j) \in \texttt{DjnAts}(R_j)$ for $j \in \{1,\ldots,k\}$ do
   for each vector $\mathcal{B} = \langle A_1,\ldots,A_k\rangle$ s.t. $A_j \in \texttt{ISubCon}[\mathcal{T}](\exists R_j)$ for $j \in \{1,\ldots,k\}$ do
   $$\mathcal{J}[\mathcal{D},\mathcal{R},\mathcal{B}] := \Big(\mathcal{J}_0 \setminus \bigcup_{i=1}^{k} \mathit{root}_{\mathcal{T}}(D_i(c_i))\Big) \cup \bigcup_{i=1}^{k} \mathit{chase}_{\mathcal{T}}(\{R_i(x_i,c_i), A_i(x_i)\}),$$
   where $\{x_1,\ldots,x_k\}$ are pairwise distinct constants from $\Delta \setminus \mathit{adom}(\mathcal{K})$ and fresh for $\mathfrak{J}$.
   $\mathfrak{J} := \mathfrak{J} \cup \{\mathcal{J}[\mathcal{D},\mathcal{R},\mathcal{B}]\}$.
3. Return $\mathfrak{J}$.

Figure 5: The procedure $\texttt{BP}(\mathcal{K},\mathcal{N})$ of building the prototypal set $\mathfrak{J}$ for a simple evolution setting $\mathcal{K} = (\mathcal{T},\mathcal{A})$ and $\mathcal{N}$

Then, it

(a) deletes from $\mathcal{J}_0$ all the atoms $D(c)$ of $\mathcal{D}$ (that is, it deletes c from $D^{\mathcal{J}_0}$) together with all the atoms that $\mathcal{T}$-entail them, (in Equations 11 and 12 of Example 27 this corresponds to

$$\mathcal{J}_0 \setminus \{\mathit{Priest}(\mathit{pedro})\} \quad \text{and} \quad \mathcal{J}_0 \setminus \{\mathit{Priest}(\mathit{ivan})\};$$

(b) adds to what remains from $\mathcal{J}_0$ the chase of pairs of atoms of the form $R(x,c)$, $A(x)$, that is, it connects with R some elements x of $\Delta \setminus \mathit{adom}(\mathcal{K})$ to the constants c; note that adding $R(x,c)$ to $\mathcal{J}_0$ before deleting $D(c)$ would violate $\mathcal{T}$ since $D(c) \in \texttt{DjnAts}(R)$. (in Equations 11 and 12 of Example 27 this respectively corresponds to adding

$$\{\mathit{HasHusb}(\_girl,\mathit{ivan})\} \cup \{\mathit{Wife}(\_girl)\} \quad \text{and} \quad \{\mathit{HasHusb}(\_girl,\_guy)\} \cup \{\mathit{Wife}(\_girl)\}.$$

Note that $\texttt{BZP}(\mathcal{K},\mathcal{N}) \subseteq \texttt{BP}(\mathcal{K},\mathcal{N})$ and $\mathcal{J}_0$ corresponds to $\mathcal{J}[\emptyset,\varepsilon,\varepsilon]$, where ε is the empty tuple.

Example 38. In Example 27, for $\mathcal{J}_1$ the set $\mathcal{D}$ is $\{\mathit{Priest}(\mathit{pedro})\}$ and for $\mathcal{J}_2$ it is $\mathcal{D} = \{\mathit{Priest}(\mathit{ivan})\}$. For $\mathcal{J}_1$ and $\mathcal{J}_2$ the vector $\mathcal{R}$ is $\langle \mathit{HasHusb}\rangle$, and the vector $\mathcal{B}$ is $\langle \mathit{Wife}\rangle$. ■

The following result establishes the fundamental property of the set of models computed by BP:

Theorem 39. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, $\mathcal{N}$ be a simple evolution setting and let $\mathcal{K}$ be in DL-Lite$_{\mathcal{R}}$. Then, the set $\texttt{BP}(\mathcal{K},\mathcal{N})$ is prototypal for $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Moreover, $|\texttt{BP}(\mathcal{K},\mathcal{N})|$ is exponential in $|\mathcal{K} \cup \mathcal{N}|$.*

Proof. To see the bound on $|\texttt{BP}(\mathcal{K},\mathcal{N})|$ observe that the number of prototypes is polynomial in the number of triples $\mathcal{D}$, $\mathcal{R}[\mathcal{D}]$ and $\mathcal{B}[\mathcal{R}]$, where the number of different components in each triple is exponential in $|\mathcal{K} \cup \mathcal{N}|$. We next exhibit models of $\mathcal{K}$ that evolve in prototypes thus showing that $\texttt{BP}(\mathcal{K},\mathcal{N}) \subseteq \mathcal{K} \diamond \mathcal{N}$.

Proposition 40. *For the zero prototype $\mathcal{J}_0$ and every other prototype $\mathcal{J}[\mathcal{D},\mathcal{R},\mathcal{B}]$ returned by $\texttt{BP}(\mathcal{K},\mathcal{N})$ it holds that $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ and $\mathcal{J}[\mathcal{D},\mathcal{R},\mathcal{B}] \in \mathcal{I}[\mathcal{D},\mathcal{R},\mathcal{B}] \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$, where $\mathcal{I}_0$ and $\mathcal{I}[\mathcal{D},\mathcal{R},\mathcal{B}]$ are models of $\mathcal{K}$ defined as following:*

$$\mathcal{I}[\mathcal{D},\mathcal{R},\mathcal{B}] = \mathcal{I}^{can} \cup \bigcup_{1 \leq i \leq |\mathcal{D}|} \mathit{chase}_{\mathcal{T}}\Big(\{R_i(x_i,d), A_i(x_i) \mid R_i \in \mathcal{R},\ A_i \in \mathcal{B},\ \mathcal{N} \models_{\mathcal{T}} \neg\exists R^-(d), \mathcal{A} \not\models_{\mathcal{T}} \neg\exists R^-(d)\}\Big). \tag{14}$$

$$\mathcal{I}_0 = \mathit{chase}_{\mathcal{T}}\Big(\mathcal{A} \cup \bigcup_{A(a)\in\mathcal{A}_1} \{R_a(a,b_a) \mid \textit{for corresponding } R_a \textit{ and } b_a\}\Big), \tag{15}$$

where the auxiliary set $\mathcal{A}_1$ is as follows:

$$\mathcal{A}_1 = \{A(a) \in cl_{\mathcal{T}}(\mathcal{A}) \mid \textit{there is } R_a \in \textit{TR}[\mathcal{K},\mathcal{N}],\ \textit{s.t. } A \in \texttt{ISubCon}[\mathcal{T}](\exists R_a) \textit{ and } \forall x \in \Delta\colon \\ \mathcal{N} \not\models_{\mathcal{T}} R_a(a,x),\ \mathcal{A} \not\models_{\mathcal{T}} R_a(a,x) \textit{ and there is } b_a \in \Delta\colon \mathcal{N} \models_{\mathcal{T}} \neg\exists R^-(b_a)\},$$

Now we show that for every model $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ there exists a model $\mathcal{J}' \in \texttt{BP}(\mathcal{K},\mathcal{N})$ such that $\mathcal{J}'$ is homomorphically embeddable into $\mathcal{J}$. We do it in three steps.

Step 1: Recall that $\mathcal{L}^a_{\subseteq}$ is local, that is, $\mathcal{K} \diamond \mathcal{N}$ is the union of $\mathcal{I} \diamond \mathcal{N}$ across all $\mathcal{I} \models \mathcal{K}$. Thus, for each $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ there is $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$ and $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$. Our first observation is that all $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ share the alignment of $\mathcal{I}$ without disjoint atoms. In terms of Example 4, these $\mathcal{J}$'s share $\mathcal{I}$ without the priests.

Proposition 41. *For every $\mathcal{I} \in Mod(\mathcal{K})$ and $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ it holds: $Align_{\mathcal{T}}(\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}, \mathcal{N}) \subseteq \mathcal{J}$, where*

$$\mathcal{B}_{\mathcal{I}} = \bigcup_{D(c) \in \mathcal{S}} root^{\mathcal{I}}_{\mathcal{T}}(D(c)),\ \textit{where}$$

$$\mathcal{S} = \{D(c) \in \mathcal{I} \mid \mathcal{N} \not\models_{\mathcal{T}} D(c),\ \mathcal{N} \not\models_{\mathcal{T}} \neg D(c),\ \textit{and there is } R \textit{ dually-affected in } \mathcal{K} \textit{ s.t.} \\ \{\exists R^-(c), D(c)\} \models_{\mathcal{T}} \bot,\ \textit{and there are } x, d \in \Delta \textit{ s.t. } \mathcal{I} \models R(x,d),\ \mathcal{N} \models \neg\exists R^-(d)\}.$$

In particular, if $\mathcal{I}$ is $\mathcal{I}^{can}$, i.e., a canonical model for $\mathcal{K}$, then for every $\mathcal{J} \in \mathcal{I}^{can} \diamond \mathcal{N}$ it holds that $Align_{\mathcal{T}}(\mathcal{I}^{can} \setminus \texttt{DjnAts}[\mathcal{K},\mathcal{N}], \mathcal{N}) \subseteq \mathcal{J}$.

Note that $\mathcal{B}_{\mathcal{I}}$ in fact extends the set of disjoint atoms $\texttt{DjnAts}[\mathcal{K},\mathcal{N}]$ from KBs to models of this KBs in the sense that $\mathcal{B}_{\mathcal{I}} \cap fcl_{\mathcal{T}}(\mathcal{A}) = \texttt{DjnAts}[\mathcal{K},\mathcal{N}]$. As a consequence of Proposition 41, for a given $\mathcal{I} \in Mod(\mathcal{K})$, every $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ can be partitioned in two parts: *(i)* a *constant* part $Align_{\mathcal{T}}(\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}, \mathcal{N})$ which is the same across all elements of $\mathcal{I} \diamond \mathcal{N}$; we denote this part as $\mathcal{J}_c$, *(ii)* and a *variable* part $\mathcal{J}_v = \mathcal{J} \setminus \mathcal{J}_c$, which may vary from one element of $\mathcal{I} \diamond \mathcal{N}$ to another. Clearly, $\mathcal{J} = \mathcal{J}_c \cup \mathcal{J}_v$ and $\mathcal{J}_c \cap \mathcal{J}_v = \emptyset$. Note that $\mathcal{J}_c$ is the constant part of the entire set $\mathcal{I} \diamond \mathcal{N}$ in the sense that $\mathcal{J}_c \subseteq \cap_{\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}} \mathcal{J}'$. In the following we will write $\mathcal{J}_c$ to denote a constant part of the set $\mathcal{I} \diamond \mathcal{N}$, where $\mathcal{I}$ is some model of $\mathcal{K}$ satisfying $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$.

Step 2: Now observe that $\mathcal{J}_c$ for $\mathcal{J} \in \texttt{BP}(\mathcal{K},\mathcal{N})$ can be homomorphically embedded in $\mathcal{J}'_c$ for every $\mathcal{J}' \in \mathcal{K} \diamond \mathcal{N}$ such that $\mathcal{J} \hookrightarrow \mathcal{J}'$.

Proposition 42. *Let $\mathcal{J} \in \texttt{BP}(\mathcal{K},\mathcal{N})$ and $\mathcal{J}' \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ such that $\mathcal{J} \hookrightarrow \mathcal{J}'$. Then, $\mathcal{J}_c \hookrightarrow \mathcal{J}'_c$.*

Step 3: Recall that our aim is to show that for every $\mathcal{J} \in \mathcal{K} \diamond \mathcal{N}$ there is $\mathcal{J}' \in \texttt{BP}(\mathcal{K},\mathcal{N})$ such that $\mathcal{J}' \hookrightarrow \mathcal{J}$, or, equivalently, $(\mathcal{J}'_c \cup \mathcal{J}'_v) \hookrightarrow (\mathcal{J}_c \cup \mathcal{J}_v)$. Due to Proposition 42, we have that $\mathcal{J}'_c \hookrightarrow \mathcal{J}_c$ for every $\mathcal{J}' \in \texttt{BP}(\mathcal{K},\mathcal{N})$, thus, it is enough to exhibit

$$\mathcal{J}' \in \texttt{BP}(\mathcal{K},\mathcal{N}) \text{ such that } \mathcal{J}'_v \hookrightarrow \mathcal{J}_v. \tag{16}$$

The following properties of models in $\mathcal{K} \diamond \mathcal{N}$ will guarantee that such a $\mathcal{J}'$ exists.

Proposition 43. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ be a simple evolution setting, $\mathcal{I} \models \mathcal{K}$, and $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. If $D(c) \in \texttt{DjnAts}[\mathcal{K},\mathcal{N}]$, then the following holds:*

(i) If $D(c) \notin \mathcal{J}$, then there exists $R \in TR$ and $A \in \texttt{ISubCon}[\mathcal{T}](\exists R)$ s.t. $D(c) \in \texttt{DjnAts}[\mathcal{K},\mathcal{N}](R)$ and

$$\{R(x,c), A(x)\} \subseteq \mathcal{J} \textit{ for some } x \in \Delta. \tag{17}$$

(ii) If $D(c) \in \mathcal{J}$, then for every unary MA $A(c)$ satisfying $\mathcal{K} \models A(c)$, where $\mathcal{T} \models A \sqsubseteq D$ and $A(c) \in \mathit{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$, the inclusion $A(c) \in \mathcal{J}$ holds.[3]

Now we have all the components to exhibit $\mathcal{J}'$ satisfying Equation 16. Let $\mathcal{J}$ be a model of $\mathcal{K} \diamond \mathcal{N}$. The relevant part of $\mathcal{J}_v$ in which we will homomorphically embed $\mathcal{J}'_v$ of Equation 16 is constructed as follows. Consider $\mathcal{D}_{\mathcal{J}}$ – the set of all (redundant) atoms from $\mathtt{DjnAts}[\mathcal{K}, \mathcal{N}]$ that are *not* in $\mathcal{J}$, that is, $\mathcal{D}_{\mathcal{J}} \subseteq \mathtt{DjnAts}[\mathcal{K}, \mathcal{N}]$ and for every $D(c) \in \mathcal{D}_{\mathcal{J}}$ we have $D(c) \notin \mathcal{J}$, while for every $D(c) \in \mathtt{DjnAts}[\mathcal{K}, \mathcal{N}] \setminus \mathcal{D}_{\mathcal{J}}$ we have $D(c) \in \mathcal{J}$. By Proposition 43, Case (i), for every $D(c) \in \mathcal{D}_{\mathcal{J}}$ there is a pair of atoms

$$\{R^{D(c)}(x, c), A^{D(c)}(x)\} \subseteq \mathcal{J}_v \tag{18}$$

where $x \in \Delta$, $R^{D(c)} \in \mathit{TR}$ and $A^{D(c)} \in \mathtt{ISubCon}[\mathcal{T}](\exists R^{D(c)})$. Observe that for each $D(c)$ there might be several such pairs of atoms. Let $\mathcal{R}_{\mathcal{J}} = \langle R^{D(c)} \mid D(c) \in \mathcal{D}_{\mathcal{J}} \rangle$ and $\mathcal{B}_{\mathcal{J}} = \langle A^{D(c)} \mid D(c) \in \mathcal{D}_{\mathcal{J}} \rangle$. Now take $\mathcal{J}'$ as the prototype $\mathcal{J}[\mathcal{D}_{\mathcal{J}}, \mathcal{R}_{\mathcal{J}}, \mathcal{B}_{\mathcal{J}}]$ constructed by $\mathtt{BP}$ in Figure 5. Observe that by the definition of $\mathtt{BP}$ the variable part of $\mathcal{J}'$ is the following:

$$\mathcal{J}[\mathcal{D}_{\mathcal{J}}, \mathcal{R}_{\mathcal{J}}, \mathcal{B}_{\mathcal{J}}]_v = \bigcup_{D(c) \in \mathcal{D}_{\mathcal{J}}} \{R^{D(c)}(x', c), A^{D(c)}(x') \mid x' \text{ is fresh}\} \cup \tag{19}$$

$$\bigcup_{D(c) \in \mathtt{DjnAts}[\mathcal{K},\mathcal{N}] \setminus \mathcal{D}_{\mathcal{J}}} \Big(\{D(c)\} \cup \{A(c) \mid \mathcal{K} \models A(c), A(c) \notin \mathcal{D}_{\mathcal{J}}, A(c) \in \mathit{Align}_{\mathcal{T}}(\mathcal{A}, \mathcal{N}), \mathcal{T} \models A \sqsubseteq D\} \Big),$$

where "x' is fresh" means that it is in $\Delta \setminus \mathit{adom}(\mathcal{K})$ and distinct for each set $\{R^{D(c)}(x', c), A^{D(c)}(x')\}$ defined by $D(c)$. Also observe that if $\mathcal{D} = \emptyset$, then $\mathcal{J}[\mathcal{D}_{\mathcal{J}}, \mathcal{R}_{\mathcal{J}}, \mathcal{B}_{\mathcal{J}}]$ is the zero prototype $\mathcal{J}_0$. Note that the constant part $\mathcal{J}'_c$ of $\mathcal{J}'$ can be constructed by applying Proposition 41 to $\mathcal{I}'$ such that $\mathcal{J}' = \mathcal{I}' \diamond \mathcal{N}$, where $\mathcal{I}'$ is as in Equation 15 if $\mathcal{D} = \emptyset$ and as is in Equation 14 otherwise.

It remains to show that there is a homomorphism h from $\mathcal{J}'_v$ to $\mathcal{J}_v$. Consider a mapping h such that it is the identity mapping on every element of $\mathcal{J}'_v$, but x' of Equation 19, and $h(x') = x$, where x is of Equation 17, for every such x'. To see that h is a homomorphism observe how it works on every atom that occurs in Equation 19: *(i)* for every $D(c) \in \mathcal{D}_{\mathcal{J}}$, that is, $D(c) \notin \mathcal{J}$ we have $R^{D(c)}(h(x'), h(c)) = R^{D(c)}(x, c)$, where $R^{D(c)}(x, c) \in \mathcal{J}_v$ due to Equation 18; and $A^{D(c)}(h(x')) = A^{D(c)}(x)$, where $A^{D(c)}(x) \in \mathcal{J}_v$ also due to Equation 18, *(ii)* for every $D(c) \in \mathtt{DjnAts} \setminus \mathcal{D}_{\mathcal{J}}$, that is, $D(c) \in \mathcal{J}$, we have $D(h(c)) = D(c)$; and for $A(c) \in \mathit{Align}_{\mathcal{T}}(\{A(c) \mid \mathcal{K} \models A(c), \mathcal{T} \models A \sqsubseteq D\}, \mathcal{N})$, we have $A(h(c)) = A(c)$, where $A(c) \in \mathcal{J}_v$ holds since $A(c)$ satisfies the conditions of Proposition 43, Case (ii). □

Extension of $\mathtt{BP}$ to DL-Lite$_{\mathcal{R}}$ KBs without restrictions. Results of Theorem 39 can be extended to the general case when the evolution setting is not simple. Observe that in the general case the $\mathtt{BP}$ procedure does return prototypes but not all of them. Weakening restrictions in Cases (i) and (iii) in the definition of simple evolution settings (i.e., allowing entailments from $\mathcal{K}$ of the form $\exists R \sqsubseteq A$ and $\mathcal{T}$-entailments from $\mathcal{N}$ of the form $\exists R(a)$) results in more than one zero prototype. Weakening in Case (ii) (i.e., allowing entailment from $\mathcal{K}$ of direct role interactions of the form $\exists R \sqsubseteq \exists R'$ and $\exists R \sqsubseteq \neg\exists R'$), leads to the need to iterate $\mathtt{BP}$ over constructed prototypes. More precisely, to gain the missing prototypes in this case one should run $\mathtt{BP}$ several times (a finite number) iterating over (already constructed) prototypes until no new prototypes can be constructed. Intuitively, the reason is that $\mathtt{BP}$ deletes disjoint atoms (atoms of $\mathtt{DjnAts}$) and adds new atoms of the form $R(a, b)$ for some triggered dually-affected role R, which may in turn trigger another

[3] Recall that the evolution setting $\mathcal{K}, \mathcal{N}$ is simple and therefore there is no role R such that $\exists R \sqsubseteq D$.

dually-affected role, say P, and such triggering may require further modifications, already for P. These further modifications require a new run of BP. For example, if we have $\exists R^- \sqsubseteq \neg\exists P^-$ in the TBox and we set $R(a, b)$ in a prototype, say $\mathcal{J}_k$, this modification triggers role P and we should run BP recursively with the prototype $\mathcal{J}_k$ as if it was the zero prototype. We shall not discuss the general procedures in more details due to space limitations.

5.3. Approximating $\mathcal{L}^a_{\subseteq}$ Evolution and Practical Considerations

Capturing Evolution Results. We start with a discussion on *how* to capture $\mathcal{L}^a_{\subseteq}$ evolution of *DL-Lite*$_\mathcal{R}$ KBs in logics richer than *DL-Lite*$_\mathcal{R}$. As we saw in the previous section, for every *DL-Lite*$_\mathcal{R}$ evolution setting $\mathcal{K}$ and $\mathcal{N}$, the evolution result $\mathcal{K} \diamond \mathcal{N}$ is a finite union of sets of models, $\mathcal{K} \diamond \mathcal{N} = \bigcup_i \mathcal{S}_i$, where each $\mathcal{S}_i$ contains a prototype $\mathcal{J}_i$. Thus, axiomatization of $\mathcal{K} \diamond \mathcal{N}$ boils down to axiomatization of each $\mathcal{S}_i$ with some theory Th_i, that is, $\mathcal{S}_i = Mod(Th_i)$, and taking the disjunction across these theories. As shown in [14], each Th_i can be computed based on a prototype of $\mathcal{S}_i$ using a *DL-Lite*$_\mathcal{R}$ KB $\mathcal{K}_i[\mathcal{J}_i]$ (whose canonical model is precisely $\mathcal{J}_i$) and a compensation formula Ψ, which is not expressible in *DL-Lite*$_\mathcal{R}$, as $Th_i \equiv \mathcal{K}_i[\mathcal{J}_i] \wedge \Psi$. It turned out [14] that Ψ is the same for each Th_i, hence:

$$\mathcal{K} \diamond \mathcal{N} = Mod(\Psi \wedge \bigvee_{i=1}^{n} \mathcal{K}[\mathcal{J}_i]),$$

We showed that $Th_{\mathcal{K}\diamond\mathcal{N}} = \Psi \wedge \bigvee_{i=1}^{n} \mathcal{K}[\mathcal{J}_i]$ is in FO[2] and even in $\mathcal{SHOIQ}$ [3].

On the one hand we do not know how to do $\mathcal{L}^a_{\subseteq}$ evolution of $\mathcal{SHOIQ}$ KBs: if one wants to do evolution of $Th_{\mathcal{K}\diamond\mathcal{N}}$ with some new knowledge, then it is still unclear which logic is needed and how to capture the evolution result. On the other hand, we would like to stay within *DL-Lite*$_\mathcal{R}$ and return a *DL-Lite*$_\mathcal{R}$ KB as the evolution result. Therefore, we study now how to *approximate* $Th_{\mathcal{K}\diamond\mathcal{N}}$ in *DL-Lite*$_\mathcal{R}$.

Approximating Evolution Results. Since neither the disjunction of $\mathcal{K}_i[\mathcal{J}_i]$ nor Ψ is expressible in *DL-Lite*$_\mathcal{R}$, one way to approximate $Th_{\mathcal{K}\diamond\mathcal{N}}$ is to take one of $\mathcal{K}_i[\mathcal{J}_i]$. Unfortunately, such an approximation is neither sound nor complete, that is, for each i there are models of $\mathcal{K}_i[\mathcal{J}_i]$ that are not in $\mathcal{K} \diamond \mathcal{N}$ and there are models of $\mathcal{K} \diamond \mathcal{N}$ that are not in $Th_{\mathcal{K}\diamond\mathcal{N}}$. What we propose next is an approximation that is sound and keeps the certain knowledge of $\mathcal{K} \diamond \mathcal{N}$, that is, ABox assertions shared by all $\mathcal{K}_i[\mathcal{J}_i]$.

We say that a membership assertion g (positive or negative) is *certain for evolution of $\mathcal{K}$ with $\mathcal{N}$* if $\mathcal{K} \diamond \mathcal{N} \models g$. Since we allow only positive MAs in $\mathcal{N}$, negative certain MAs can come from $\mathcal{K}$ only. The next lemma shows that negative MAs are certain when they are in the alignment of $\mathcal{K}$ with $\mathcal{N}$, that is, they do $\mathcal{T}$-contradict $\mathcal{N}$.

Lemma 44. *Let $\mathcal{K}$ and $\mathcal{N}$ be a simple evolution setting, $\mathcal{K}$ be in DL-Lite$_\mathcal{R}$, and g a negative membership assertion. Then $\mathcal{K} \diamond \mathcal{N} \models g$ if and only if*

$$g \in fcl_\mathcal{T}(\mathcal{N}) \cup \left(\mathtt{AlignAlg}(\mathcal{K},\mathcal{N}) \setminus \bigcup_{R \in TR} \bigcup_{A(c) \in \mathtt{DjnAts}[\mathcal{K},\mathcal{N}](R)} \neg\exists R^-(c) \right).$$

Proof. First we show the "only if" direction. Suppose that $g \in fcl_\mathcal{T}(\mathcal{N}) \cup CertainNegative$, where $CertainNegative = \mathtt{AlignAlg}(\mathcal{K},\mathcal{N}) \setminus \bigcup_{R \in TR} \bigcup_{A(c) \in \mathtt{DjnAts}[\mathcal{K},\mathcal{N}](R)} \neg\exists R^-(c)$. If $g \in fcl_\mathcal{T}(\mathcal{N})$ then, clearly, by the definition of $\mathcal{K} \diamond \mathcal{N}$, g is certain. Suppose that $g \in CertainNegative \setminus fcl_\mathcal{T}(\mathcal{N})$. Assume g is not certain, that is, there is a model $\mathcal{J}_0 \in \mathcal{K} \diamond \mathcal{N}$ such that $\mathcal{J}_0 \models \neg g$. Let $\mathcal{I}_0$ be a model of $\mathcal{K}$ such that $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$. Then we have the following cases:

1. g is of the form $\neg A(c)$, that is, $A(c) \in \mathcal{J}_0$. It is easy to see that in this case $\mathcal{N} \parallel_{\mathcal{T}} A(c)$, and consequently the model $\mathcal{J}'_0 = \mathcal{J}_0 \setminus \mathit{root}_{\mathcal{T}} A(c)$ is in $\mathit{Mod}(\mathcal{N})$. Also $\mathcal{J}'_0$ is in $\mathit{Mod}(\mathcal{T})$ due to Proposition 16. It is easy to check that by definition of $\mathcal{J}'_0$ and due to restrictions of simple evolution settings, the following holds: $\mathcal{I}_0 \ominus \mathcal{J}'_0 \subseteq \mathcal{I}_0 \ominus \mathcal{J}_0$ and $A(c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I}_0 \ominus \mathcal{J}'_0)$. Thus, we obtain a contradiction with $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$.
2. g is of the form $\neg \exists R^-(c)$, that is, there exists $\alpha \in \Delta$ such that $R(\alpha, c) \in \mathcal{J}_0$. Due to restrictions of simple evolution settings, we have that neither $\mathcal{N} \models_{\mathcal{T}} \exists R(\alpha)$ nor $\mathcal{N} \models_{\mathcal{T}} \exists R^-(c)$ holds. This means that the model $\mathcal{J}'_0 = \mathcal{J}_0 \setminus \mathit{root}_{\mathcal{T}}(\exists R(\alpha)) \setminus \mathit{root}_{\mathcal{T}}(\exists R^-(c))$ is a model of $\mathcal{N}$. Also, due to Proposition 16, we have that $\mathcal{J}'_0$ is in $\mathit{Mod}(\mathcal{T})$. It is easy to check that $\mathcal{I}_0 \ominus \mathcal{J}'_0 \subseteq \mathcal{I}_0 \ominus \mathcal{J}_0$ and $R(\alpha, c) \in (\mathcal{I}_0 \ominus \mathcal{J}_0) \setminus (\mathcal{I}_0 \ominus \mathcal{J}'_0)$. Thus, we obtain a contradiction with $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$.

Therefore, if $g \in \mathit{fcl}_{\mathcal{T}}(\mathcal{N}) \cup \mathit{CertainNegative}$, then $\mathcal{K} \diamond \mathcal{N} \models g$.

We show now the "if" direction. Suppose that $\mathcal{K} \diamond \mathcal{N} \models g$, but $g \notin \mathit{fcl}_{\mathcal{T}}(\mathcal{N}) \cup \mathit{CertainNegative}$. There are two possible cases. If $g \in \mathtt{AlignAlg}(\mathcal{K},\mathcal{N})$ and $g \in \bigcup_{R \in TR} \bigcup_{A(c) \in \mathtt{DjnAts}[\mathcal{K},\mathcal{N}](R)} \neg \exists R^-(c)$, then there exists a concept B such that $\mathcal{T} \models \exists R^- \sqsubseteq \neg B$ and $\mathcal{A} \models_{\mathcal{T}} B(c)$. It is easy to check that the prototype $\mathcal{J}[\{B(c)\}, \langle R \rangle, \langle A \rangle]$ for some $A \in \mathtt{ISubCon}(R)$ is such that it does not satisfy $\neg \exists R^-(c)$. We obtain a contradiction with the assumption that g is certain, and therefore $\mathcal{K} \diamond \mathcal{N} \not\models g$. Finally, suppose that $g = \neg f \notin \mathit{fcl}_{\mathcal{T}}(\mathcal{N}) \cup \mathit{CertainNegative}$, which leads to $\mathcal{N} \parallel_{\mathcal{T}} f$ and $\mathcal{A} \parallel_{\mathcal{T}} f$. Assume that $\mathcal{K} \diamond \mathcal{N} \models g$. Consider models $\mathcal{J}_0 \in \mathcal{K} \diamond \mathcal{N}$ and $\mathcal{I}_0 \in \mathit{Mod}(\mathcal{K})$ such that $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$. Then consider a set $\mathcal{S} = \mathit{chase}_{\mathcal{T}}(f)$, where all the Δ-elements are such that they do not occur in $\mathit{adom}(\mathcal{K} \cup \mathcal{N})$, and models $\mathcal{I}_0$ and $\mathcal{J}_0$. Using $\mathcal{S}$ we define the following two models: $\mathcal{I}'_0 = \mathcal{I}_0 \cup \mathcal{S}$ and $\mathcal{J}'_0 = \mathcal{J}_0 \cup \mathcal{S}$. It is easy to check that $\mathcal{I}'_0 \in \mathit{Mod}(\mathcal{T} \cup \mathcal{A})$ and $\mathcal{J}_0 \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$. We are going to show now that $\mathcal{J}'_0 \in \mathcal{I}'_0 \diamond \mathcal{N}$, which will lead to a contradiction with the fact that $\mathcal{K} \diamond \mathcal{N} \models g$ since $\mathcal{J}'_0 \not\models g$. Suppose that $\mathcal{J}'_0 \notin \mathcal{I}'_0 \diamond \mathcal{N}$, that is, there exists a model $\mathcal{J}''_0 \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $\mathcal{I}'_0 \ominus \mathcal{J}''_0 \subsetneq \mathcal{I}'_0 \ominus \mathcal{J}'_0$. Thus, there is an atom $f \in \mathcal{J}'_0$ such that $f \in (\mathcal{I}'_0 \ominus \mathcal{J}'_0) \setminus (\mathcal{I}'_0 \setminus \mathcal{J}''_0)$. Note that $f \notin \mathcal{S}$ since $\mathcal{S} \not\subseteq \mathcal{I}'_0 \ominus \mathcal{J}'_0$, and also $\mathcal{J}'' = \mathcal{J}''_0 \setminus (\mathcal{S} \setminus \mathcal{J}_0)$ is in $\mathit{Mod}(\mathcal{T} \cup \mathcal{N})$. These two observations lead to the fact that $\mathcal{I}_0 \ominus \mathcal{J}'' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$ which contradicts with $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$. Therefore, $\mathcal{J}'_0 \in \mathcal{I}'_0 \diamond \mathcal{N}$. Since $\mathcal{J}'_0 \not\models g$ we conclude that g is not certain. □

The next lemma shows that positive MAs are certain when they are satisfied by each prototype of $\mathcal{K} \diamond \mathcal{N}$.

Lemma 45. *Let $\mathcal{K}$ and $\mathcal{N}$ be a simple evolution setting, $\mathcal{K}$ be in DL-Lite$_{\mathcal{R}}$, g a positive membership assertion, and $\mathfrak{J}$ a prototypal set for $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Then $\mathcal{K} \diamond \mathcal{N} \models g$ if and only if $\bigcap_{\mathcal{J} \in \mathfrak{J}} \mathcal{J} \models g$.*

Proof. First we show the "if" direction. Suppose that a positive MA g is certain for evolution of $\mathcal{K}$ with $\mathcal{N}$, i.e., $\mathcal{K} \diamond \mathcal{N} \models g$. In particular, this means that for every prototype $\mathcal{J} \in \mathfrak{J}$, it holds that $\mathcal{J} \models g$. If g is of the form $A(b)$ or $R(b, c)$, then $g \in \mathcal{J}$, and if g is of the form $\exists R(b)$, then there exists an element $\alpha \in \Delta$ such that $R(b, \alpha) \in \mathcal{J}$. In both cases we have that $\bigcap_{\mathcal{J} \in \mathfrak{J}} \mathcal{J} \models g$.

We now show the "only if" direction. Suppose that g is a positive MA such that $\bigcap_{\mathcal{J} \in \mathfrak{J}} \mathcal{J} \models g$. Let $\mathcal{J}_0$ be in $\mathcal{K} \diamond \mathcal{N}$. Consider a prototype $\mathcal{J}'_0$ in $\mathfrak{J}$ for which there exists a homomorphism h such that $h : \mathcal{J}'_0 \hookrightarrow \mathcal{J}_0$. Since $\mathcal{J}'_0 \models g$, we have three possibilities:

(i) If g is of the form $A(b)$, then $A(b) \in \mathcal{J}'_0$ and hence $A(h(b)) = A(b) \in \mathcal{J}_0$.
(ii) If g is of the form $R(b, c)$, then, analogously to the previous case, $R(b, c) \in \mathcal{J}_0$.
(iii) If g is of the form $\exists R(b)$, then there exists an element $\alpha \in \Delta$ such that $R(b, \alpha) \in \mathcal{J}'_0$ and therefore $R(h(b), h(\alpha)) = R(b, h(\alpha)) \in \mathcal{J}_0$.

In all the three cases we have that $\mathcal{J}_0 \models g$ which concludes the proof. □

Based on the preceding lemmas, we conclude with an algorithm to compute a maximal sound approximation.

Theorem 46. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ be a simple evolution setting and $\mathcal{K}$ be in DL-Lite$_\mathcal{R}$. Then the DL-Lite$_\mathcal{R}$ KB $\mathcal{K}^{app} = \mathcal{T} \cup \mathcal{A}^{app}$, where*

$$\mathcal{A}^{app} := \mathcal{N} \cup \left(\texttt{AlignAlg}(\mathcal{K},\mathcal{N}) \setminus \bigcup_{R \in TR} \bigcup_{A(c) \in \texttt{DjnAts}[\mathcal{K},\mathcal{N}](R)} \neg\exists R^-(c) \setminus \mathcal{A}^{aux} \right), \text{ and}$$

$$\mathcal{A}^{aux} = \bigcup_{R \in TR[\mathcal{T},\mathcal{N}]} \bigcup_{\substack{a \in adom(\mathcal{K}\cup\mathcal{N}) \text{ s.t.} \\ R(a,b) \notin \texttt{AlignAlg}(\mathcal{K},\mathcal{N}) \cup \mathcal{N} \\ \text{for every } b \in adom(\mathcal{K}\cup\mathcal{N}):}} root_\mathcal{T}(\exists R(a)) \cup \bigcup_{g \in \texttt{DjnAts}[\mathcal{K},\mathcal{N}]} root_\mathcal{T}(g)$$

can be computed in polynomial time in $|\mathcal{K} \cup \mathcal{N}|$ and is a maximal sound approximation of $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$.

Proof. The proof follows from Lemma 44, and Lemma 45 coupled with the correctness of the the BP procedure. □

As a corollary of the theorem above, consider the case when there are no triggered roles for $\mathcal{K}$ and $\mathcal{N}$. Observe that in this case $\texttt{DjnAts}[\mathcal{K},\mathcal{N}] = \emptyset$ and $TR[\mathcal{T},\mathcal{N}] = \emptyset$. The former condition implies $\cup_{g \in \texttt{DjnAts}[\mathcal{K},\mathcal{N}]}(root_\mathcal{T}(g) \cup root_\mathcal{T}(\neg g)) = \emptyset$, while the latter one implies $\mathcal{A}^{aux} = \emptyset$, thus yielding that $\mathcal{A}^{app} = \mathcal{N} \cup \texttt{AlignAlg}(\mathcal{K},\mathcal{N})$. We conclude that $\mathcal{K}^{app} = \mathcal{T} \cup \mathcal{N} \cup \texttt{AlignAlg}(\mathcal{K},\mathcal{N})$ is a maximal sound approximation (when $\mathcal{K}$ and $\mathcal{N}$ have no triggered roles), while we do not know whether it is also a complete approximation. Note that for *DL-Lite*$^{pr}_\mathcal{R}$ we proved (Theorem 12) that $\mathcal{T} \cup \mathcal{N} \cup \texttt{AlignAlg}(\mathcal{K},\mathcal{N})$, which coincides with $\mathcal{K}^{app}$, is an exact (sound and complete) approximation of $\mathcal{K} \diamond \mathcal{N}$. Thus, completeness of $\mathcal{K}^{app}$ would imply that Theorem 46 gives an alternative proof for Theorem 12 and extends it to a fragment of *DL-Lite*$_\mathcal{R}$ that is wider than *DL-Lite*$^{pr}_\mathcal{R}$.

Example 47. Continuing with Example 27, we compute the approximation of $\mathcal{K}^{ex}_2 \diamond \mathcal{N}^{ex}_1$. First, we compute the necessary components $fcl_\mathcal{T}(\mathcal{A}^{ex}_2)$, $\texttt{AlignAlg}(\mathcal{K}^{ex}_2, \mathcal{N}^{ex}_1)$, and $\mathcal{A}^{aux}$: one can see that $\mathcal{A}^{aux} = \emptyset$,

$$fcl_\mathcal{T}(\mathcal{A}^{ex}_2) = \mathcal{A}^{ex}_2 \cup \{\neg Priest(john),\ \neg\exists HasHusb^-(pedro),\ \neg\exists HasHusb^-(ivan)\},$$

$$\texttt{AlignAlg}(\mathcal{K}^{ex}_2, \mathcal{N}^{ex}_1) = \{Priest(pedro),\ Priest(ivan), \neg\exists HasHusb^-(pedro),\ \neg\exists HasHusb^-(ivan)\}.$$

Next we compute the set of negative MAs that are *not* certain for evolution of $\mathcal{K}^{ex}_2$ with $\mathcal{N}^{ex}_1$ (see Lemma 44). It turns out that this set is equal to $\texttt{AlignAlg}(\mathcal{K}^{ex}_2, \mathcal{N}^{ex}_1)$:

$$\bigcup_{g \in \texttt{DjnAts}[\mathcal{K},\mathcal{N}]} (root_\mathcal{T}(g) \cup root_\mathcal{T}(\neg g)) = \texttt{AlignAlg}(\mathcal{K}^{ex}_2, \mathcal{N}^{ex}_1).$$

This is not surprising since none of the MAs in $\texttt{AlignAlg}(\mathcal{K}^{ex}_2, \mathcal{N}^{ex}_1)$ holds in every prototype. For example, consider pairs of literals $Priest(pedro)$ with $\neg\exists HasHusb^-(pedro)$ and $Priest(ivan)$ with $\neg\exists HasHusb^-(ivan)$. The former pair does not hold in $\mathcal{J}_1$, and the latter one in $\mathcal{J}_2$, while both pairs hold in $\mathcal{J}_0$. Finally, we compute $\mathcal{A}^{app}$ (see Theorem 46):

$$\mathcal{A}^{app} = \mathcal{N}^{ex}_1 \cup (\texttt{AlignAlg}(\mathcal{K}^{ex}_2, \mathcal{N}^{ex}_1) \setminus \texttt{AlignAlg}(\mathcal{K}^{ex}_2, \mathcal{N}^{ex}_1) \setminus \mathcal{A}^{aux}) = \mathcal{N}^{ex}_1 = \{Priest(john)\}.$$

Therefore, the minimal sound *DL-Lite*$_\mathcal{R}$-approximation of $\mathcal{K}^{ex}_2 \diamond \mathcal{N}^{ex}_1$ under $\mathcal{L}^a_\subseteq$ is $\mathcal{K}^{app} = (\mathcal{T}^{ex}_2, \mathcal{N}^{ex}_1)$.

If we consider the evolution of $\mathcal{K}_3^{ex} = (\mathcal{T}_2^{ex}, \mathcal{A}_3^{ex})$ with $\mathcal{N}_1^{ex}$, where $\mathcal{A}_3^{ex} = \mathcal{A}_2^{ex} \cup \{ \mathit{Wife}(\mathit{mary})\}$, then

$$\texttt{AlignAlg}(\mathcal{A}_3^{ex}) = \texttt{AlignAlg}(\mathcal{A}_2^{ex}) \cup \{ \mathit{Wife}(\mathit{mary}), \exists \mathit{HasHusb}(\mathit{mary})\}.$$

Observe that neither $\mathit{Wife}(\mathit{mary})$ nor $\exists \mathit{HasHusb}(\mathit{mary})$ is in $\cap_{\mathcal{J} \in \mathfrak{J}} \mathcal{J}$ (for example, they do not hold in $\mathcal{J}_0$), and due to Lemma 45 they are uncertain and should not be present in the approximation. It is easy to check that these two assertions form $\mathcal{A}^{aux}$, i.e.,

$$\mathcal{A}^{aux} = \{ \mathit{Wife}(\mathit{mary}), \exists \mathit{HasHusb}(\mathit{mary})\}.$$

Therefore, the approximation $\mathcal{A}^{app}$ is as again:

$$\mathcal{A}^{app} = \mathcal{N}_1^{ex} \cup (\texttt{AlignAlg}(\mathcal{K}_3^{ex}, \mathcal{N}_1^{ex}) \setminus \texttt{AlignAlg}(\mathcal{K}_2^{ex}, \mathcal{N}_1^{ex}) \setminus \mathcal{A}^{aux}) = \mathcal{N}_1^{ex} = \{\mathit{Priest}(\mathit{john})\}.$$

To sum up, as soon as a husband john, who is married to some unknown individual decides to become a priest, the algorithm that computes maximal-sound approximation forces us to delete all the priests and all the wives from the old knowledge. The reason is that we do not know who of the wives from the old knowledge were married to john and who are their new husbands: either some of the former priest, or even no one. To account for this uncertainty, the atoms about wives and priests should be erased from the old KB. Thus, maximal-sound approximation $\mathcal{K}^{app}$ may erase a lot of old knowledge and the approximation result may be quite unexpected from the user point of view. ■

6. Practical Summary and Conclusion

Practical Summary. We summarize here on how one can use the results of this paper to do ABox evolution of *DL-Lite*$_\mathcal{R}$ KBs in practice. Given a *DL-Lite*$_\mathcal{R}$ evolution setting $\mathcal{K}$ and $\mathcal{N}$, one can first check (in polynomial time) whether $\mathcal{K}$ is in *DL-Lite*$_\mathcal{R}^{pr}$. If this is the case, then one can compute in polynomial time an axiomatization of the evolution result $\mathcal{K} \diamond \mathcal{N}$ under six out of eight model-based semantics using the techniques of Theorems 12 and 22. One can also compute its maximal sound approximation under the remaining two semantics $\mathcal{L}_\subseteq^s$ and $\mathcal{L}_\#^s$ using the techniques of Theorem 25. The choice of evolution semantics for *DL-Lite*$_\mathcal{R}^{pr}$ is up to the user, while we believe that atom-based semantics behave more intuitively. For the case when $\mathcal{K}$ is *not* in *DL-Lite*$_\mathcal{R}^{pr}$, then in general the result of evolution $\mathcal{K} \diamond \mathcal{N}$ is not axiomatizable in *DL-Lite*$_\mathcal{R}$ for each of the eight MBAs [8, 9]. At the same time, for $\mathcal{L}_\subseteq^a$ one can compute in polynomial time a maximal sound approximation of $\mathcal{K} \diamond \mathcal{N}$ using the techniques of Theorem 46.

Conclusion. We studied model-based approaches to ABox evolution (update and revision) over *DL-Lite*$_{\mathcal{FR}}$ and its fragments *DL-Lite*$_\mathcal{R}$ and *DL-Lite*$_\mathcal{R}^{pr}$, which all extend (the first-order fragment of) RDFS. *DL-Lite*$_\mathcal{R}^{pr}$ is closed under most of the MBAs, while *DL-Lite*$_\mathcal{R}$ is *not* closed under any of them. We showed that if the TBox of $\mathcal{K}$ entails a pair of assertions of the form $A \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq \neg C$, then an interplay of $\mathcal{N}$ and $\mathcal{A}$ may lead to inexpressibility of $\mathcal{K} \diamond \mathcal{N}$. For *DL-Lite*$_\mathcal{R}^{pr}$ we provided algorithms to compute evolution results for six model-based approaches and approximate evolution for the remaining two. For *DL-Lite*$_\mathcal{R}$ we studied the properties of evolution under a local model-based approach $\mathcal{L}_\subseteq^a$. In particular, we introduced the notion of prototypal sets that extends the notion of canonical models. We proved that prototypal sets for $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}_\subseteq^a$ exist, and that they are of exponential size in $|\mathcal{K} \cup \mathcal{N}|$, and showed an abstract procedure that constructs them. Based on the insights gained, we proposed a polynomial time algorithm to compute a maximal sound approximation of $\mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}_\subseteq^a$. We also believe that prototypes are important since they can be used to study evolution for ontology languages other than *DL-Lite*$_\mathcal{R}$. In general, we provided some understanding on why *DL-Lite* is not closed under MBAs to evolution, and what are the properties of sets of models $\mathcal{K} \diamond \mathcal{N}$. This understanding is a prerequisite to proceed with the study of evolution in more expressive DLs and to understand what to expect from MBAs in such logics.

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Appendix A. Proofs for Section 3

Proof of Theorem 6.

It remains to show the case $\mathcal{G}^s_{\#} \preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$. Consider $\mathcal{E}_{\#} = \mathcal{K} \diamond \mathcal{N}$ wrt $\mathcal{G}^s_{\#}$, which is based on the distance $dist^s_{\#}$, and $\mathcal{E}_{\subseteq} = \mathcal{K} \diamond \mathcal{N}$ wrt $\mathcal{G}^s_{\subseteq}$, which is based on $dist^s_{\subseteq}$. We now are interested in establishing whether $\mathcal{E}_{\#} \subseteq \mathcal{E}_{\subseteq}$ holds. Assume $\mathcal{J}' \in \mathcal{E}_{\#}$ and $\mathcal{J}' \notin \mathcal{E}_{\subseteq}$. From the former assumption we conclude existence of a model $\mathcal{I}'$ such that for every pair of models $\mathcal{I} \in Mod(\mathcal{K})$ and $\mathcal{J} \in Mod(\mathcal{T} \cup \mathcal{N})$, it holds that $|\mathcal{I}_0 \ominus \mathcal{J}_0| \leq |\mathcal{I} \ominus \mathcal{J}|$. From the latter assumption, $\mathcal{J}' \notin \mathcal{E}_{\subseteq}$, we conclude existence of models $\mathcal{I}'' \in Mod(\mathcal{K})$ and $\mathcal{J}'' \in Mod(\mathcal{T} \cup \mathcal{N})$ such that $dist^s_{\subseteq}(\mathcal{I}'', \mathcal{J}'') \subsetneq dist^s_{\subseteq}(\mathcal{I}', \mathcal{J}')$. Since the signature of $\mathcal{K} \cup \mathcal{N}$ is finite, the distance $dist^s_{\subseteq}$ between every two models over this signature is also finite. Thus, we obtain that $dist^s_{\#}(\mathcal{I}'', \mathcal{J}'') \lneq dist^s_{\#}(\mathcal{I}', \mathcal{J}')$, which contradicts the fact that $\mathcal{J}' \in \mathcal{E}_{\#}$ and concludes the proof. □

Recall that if $\mathcal{J}_0 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\#}$, then there exists a model $\mathcal{I}_0 \in Mod(\mathcal{K})$ such that for every pair of models $\mathcal{I}' \in Mod(\mathcal{K})$ and $\mathcal{J}' \in Mod(\mathcal{T} \cup \mathcal{N})$, it holds that $|\mathcal{I}_0 \ominus \mathcal{J}_0| \leq |\mathcal{I}' \ominus \mathcal{J}'|$. We say that this $\mathcal{I}_0$ is *$\mathcal{G}^a_{\#}$-minimally distant form* $\mathcal{J}_0$. In order to prove Theorem 7 we need the following lemma about a property of these $\mathcal{I}_0$ and $\mathcal{J}_0$.

Lemma 48. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and $\mathcal{N}$ be an evolution setting and $\mathcal{K}$ in DL-Lite$_{\mathcal{FR}}$. Let $\mathcal{J}_0 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\#}$ and $\mathcal{I}_0$ is $\mathcal{G}^a_{\#}$-minimally distant from $\mathcal{J}_0$. Then, $|\mathcal{I}_0 \ominus \mathcal{J}_0| < \omega$.*

Proof. Suppose that $|\mathcal{I}_0 \ominus \mathcal{J}_0| = \omega$. If there exist models $\mathcal{I}' \in Mod(\mathcal{K})$ and $\mathcal{J}' \in Mod(\mathcal{T} \cup \mathcal{N})$ such that $|\mathcal{I}' \ominus \mathcal{J}'| < \omega$, this will contradict the fact that $\mathcal{J}_0 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\#}$ since $|\mathcal{I}' \ominus \mathcal{J}'| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. Now we show that these $\mathcal{I}'$ and $\mathcal{J}'$ always exist. Let $\mathcal{I}^{can}$ and $\mathcal{J}^{can}$ be canonical models of $\mathcal{K}$ and $\mathcal{T} \cup \mathcal{N}$, respectively, such that the elements of $\Delta \setminus adom(\mathcal{K} \cup \mathcal{N})$ that occur in $\mathcal{I}^{can}$ do not occur in $\mathcal{J}^{can}$ and vice-versa. Note that this is always possible due to infiniteness of Δ. For a model $\mathcal{I}$, let $\mathcal{S}(\mathcal{I})$ be the set of atoms of $\mathcal{I}$ that include at least on element of $adom(\mathcal{K} \cup \mathcal{N})$:

$$\mathcal{S}(\mathcal{I}) = \{A(x) \in \mathcal{I} \mid x \in adom(\mathcal{K} \cup \mathcal{N})\} \cup \{R(x,y) \in \mathcal{I} \mid x \in adom(\mathcal{K} \cup \mathcal{N}) \text{ or } y \in adom(\mathcal{K} \cup \mathcal{N})\}.$$

Then, we define $\mathcal{I}'$ and $\mathcal{J}'$ as follows:

$$\mathcal{I}' = \mathcal{I}^{can} \cup (\mathcal{J}^{can} \setminus \bigcup_{g \in \mathcal{S}(\mathcal{J}^{can})} root^{\mathcal{J}^{can}}_{\mathcal{T}}(g)), \quad \mathcal{J}' = \mathcal{J}^{can} \cup (\mathcal{I}^{can} \setminus \bigcup_{g \in \mathcal{S}(\mathcal{I}^{can})} root^{\mathcal{I}^{can}}_{\mathcal{T}}(g)).$$

It is easy to see that $\mathcal{I}^{can} \in Mod(\mathcal{K})$ and $\mathcal{J}^{can} \in Mod(\mathcal{T} \cup \mathcal{N})$ due to Propositions 15 and 16. Then, $\mathcal{I}' \ominus \mathcal{J}' = \bigcup_{g \in \mathcal{S}(\mathcal{J}^{can})} root^{\mathcal{J}^{can}}_{\mathcal{T}}(g)) \cup \bigcup_{g \in \mathcal{S}(\mathcal{I}^{can})} root^{\mathcal{I}^{can}}_{\mathcal{T}}(g))$. Since both $\mathcal{S}(\mathcal{I}^{can})$ and $\mathcal{S}(\mathcal{J}^{can})$ are finite and for each $g \in \mathcal{S}(\mathcal{I}^{can}) \cup \mathcal{S}(\mathcal{J}^{can})$ the set $root^{\mathcal{J}^{can}}_{\mathcal{T}}(g)$ is finite, it holds that $|\mathcal{I}' \ominus \mathcal{J}'| < \omega$. The last inequality concludes the proof. □

Proof of Theorem 7.

First we show that $\mathcal{G}^a_{\#} \preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$. Let $\mathcal{J}_0 \in Mod(\mathcal{T} \cup \mathcal{N})$. Assume $\mathcal{J}_0 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\#}$, but $\mathcal{J}_0 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\subseteq}$. The former assumption implies that there is a model $\mathcal{I}_0 \in Mod(\mathcal{K})$ that is $\mathcal{G}^a_{\#}$-minimally distant from $\mathcal{J}_0$. The latter assumption implies that there exist models $\mathcal{I}' \in Mod(\mathcal{K})$ and $\mathcal{J}' \in Mod(\mathcal{T} \cup \mathcal{N})$ such that

$$\mathcal{I}' \ominus \mathcal{J}' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0 \tag{A.1}$$

Due to Lemma 48, $|\mathcal{I}_0 \ominus \mathcal{J}_0| < \omega$, and this together with (A.1) yields that $|\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I}_0 \ominus \mathcal{J}_0|$, which contradicts the fact that $\mathcal{J}_0 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\#}$. The contradiction proves that $\mathcal{G}^a_{\#} \preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$.

We next show the cases when in *DL-Lite*$_{\mathcal{FR}}$ there is no $\preccurlyeq_{sem}$-relation between model-based semantics. We start with semantics which are based on atom. Since there are 4 atom-based semantics, there are 12 pairs of different semantics. Theorem 6 establishes $\preccurlyeq_{sem}$ relation in 4 of these pairs. Thus, we now show that this relation does not hold for the remaining 8 pairs.

Case A1: $\mathcal{L}^a_{\#} \not\preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$. Assume that $\mathcal{L}^a_{\#} \preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$. Then, due to the fact that $\mathcal{G}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$ (see Theorem 6), and transitivity of $\preccurlyeq_{sem}$-relation, we conclude that $\mathcal{L}^a_{\#} \preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$, which gives a contradiction with Case A8.

Case A2: $\mathcal{L}^a_{\#} \not\preccurlyeq_{sem} \mathcal{G}^a_{\#}$. Assume that $\mathcal{L}^a_{\#} \preccurlyeq_{sem} \mathcal{G}^a_{\#}$. Then, due to the fact that $\mathcal{G}^a_{\#} \preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$, $\mathcal{G}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$ (see Theorem 6), and transitivity of $\preccurlyeq_{sem}$-relation, we conclude that $\mathcal{L}^a_{\#} \preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$, which gives a contradiction with Case A8.

Case A3: $\mathcal{L}^a_{\subseteq} \not\preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$. Consider the following KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and a new information $\mathcal{N}$:

$$\mathcal{T} = \{A \sqsubseteq \exists R, \exists R^- \sqsubseteq \neg B\}, \qquad \mathcal{A} = \{B(c), R(a,b)\}, \qquad \mathcal{N} = \{B(b)\}.$$

Consider models $\mathcal{I}_0, \mathcal{I}_1 \in Mod(\mathcal{K})$ and models $\mathcal{J}_0, \mathcal{J}_1 \in Mod(\mathcal{T} \cup \mathcal{N})$:

$$\mathcal{I}_0 = \{B(c), R(a,b)\}, \qquad \mathcal{I}_1 = \{A(a), B(c), R(a,b)\},$$
$$\mathcal{J}_0 = \{B(c), B(b)\}, \qquad \mathcal{J}_1 = \{A(a), B(b), R(a,c)\}.$$

We now show that $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ but $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\subseteq}$.

Observe that $\mathcal{J}_1 \in \mathcal{I}_1 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$, and therefore $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$. Indeed, suppose that $\mathcal{J}_1 \notin \mathcal{I}_1 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Thus, there exists a model $\mathcal{J}_1'$ such that

$$\mathcal{I}_1 \ominus \mathcal{J}_1' \subsetneq \mathcal{I}_1 \ominus \mathcal{J}_1 = \{R(a,b), R(a,c), B(b), B(c)\}.$$

Observe that $\{B(b), R(a,b)\} \subseteq \mathcal{I}_1 \ominus \mathcal{J}_1'$. Indeed, $\mathcal{T} \cup \mathcal{N} \models \neg\exists x.R(x,b)$, implies $\mathcal{J}_1' \not\models R(a,b)$, and consequently $R(a,b) \in \mathcal{I}_1 \ominus \mathcal{J}_1'$. Then, $\mathcal{N} \models B(b)$ implies $\mathcal{J}_1' \models B(b)$ and consequently $B(b) \in \mathcal{I}_1 \ominus \mathcal{J}_1'$. Hence, at least one of $R(a,c)$ and $B(c)$ does not belong to $\mathcal{I}_1 \ominus \mathcal{J}_1'$.

- Assume that $R(a,c) \notin \mathcal{I}_1 \ominus \mathcal{J}_1'$. From $A(a) \notin \mathcal{I}_1 \ominus \mathcal{J}_1'$ and $A(a) \in \mathcal{I}_1$ we conclude $A(a) \in \mathcal{J}_1'$. Thus, there is an element $x \in \Delta$ such that $R(a,x) \in \mathcal{J}_1'$. Due to the assumption, $x \neq c$. From $\mathcal{J}_1' \not\models R(a,b)$ we conclude $x \neq b$. Thus, $R(a,x) \in \mathcal{I}_1 \ominus \mathcal{J}_1'$ while $R(a,x) \notin \mathcal{I}_1 \ominus \mathcal{J}_1$, and therefore $\mathcal{I}_1 \ominus \mathcal{J}_1' \not\subset \mathcal{I}_1 \ominus \mathcal{J}_1$, which gives a contradiction.
- Assume that $B(c) \notin \mathcal{I}_1 \ominus \mathcal{J}_1'$. Since $B(c) \in \mathcal{I}_1$, we conclude that $B(c) \in \mathcal{J}_1'$. Hence, $R(a,c) \notin \mathcal{J}_1'$, otherwise $\mathcal{J}_1' \notin Mod(\mathcal{T})$. Observe that $R(a,c) \notin \mathcal{I}_1$ and we obtain that $R(a,c) \notin \mathcal{I}_1 \ominus \mathcal{J}_1'$, which gives a contradiction as was shown in the previous case.

Thus, there is no model $\mathcal{J}_1' \in Mod(\mathcal{T} \cup \mathcal{N})$ such that $\mathcal{I}_1 \ominus \mathcal{J}_1' \subsetneq \mathcal{I}_1 \ominus \mathcal{J}_1$, and $\mathcal{J}_1 \in \mathcal{I}_1 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$.

Now we show that $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\subseteq}$. It is easy to see that for every model $\mathcal{I} \in Mod(\mathcal{K})$ the following holds:

$$\{B(b), B(c), R(a,b), R(a,c)\} \subseteq \mathcal{I} \ominus \mathcal{J}_1. \tag{A.2}$$

At the same time, $\mathcal{I}_0 \ominus \mathcal{J}_0 = \{R(a,b), B(b)\}$, which together with (A.2) yields that: for every model $\mathcal{I} \in Mod(\mathcal{K})$ we have $\mathcal{I}_0 \ominus \mathcal{J}_0 \subsetneq \mathcal{I} \ominus \mathcal{J}_1$. That is, $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\subseteq}$, and therefore $\mathcal{L}^a_{\subseteq} \not\preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$.

Case A4: $\mathcal{L}^a_{\subseteq} \not\preccurlyeq_{sem} \mathcal{G}^a_{\#}$. Assume that $\mathcal{L}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^a_{\#}$. Then, due to the fact that $\mathcal{G}^a_{\#} \preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$, $\mathcal{G}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{L}^a_{\subseteq}$ (see Theorem 6) and transitivity of $\preccurlyeq_{sem}$-relation, we conclude that $\mathcal{L}^a_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^a_{\subseteq}$, which gives a contradiction with Case A3.

Case A5: $\mathcal{L}^a_{\subseteq} \not\leq_{sem} \mathcal{L}^a_{\#}$. Consider the following KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and a new information $\mathcal{N}$:

$$\mathcal{T} = \{A \sqsubseteq \exists R, \exists R^- \sqsubseteq \neg B\}, \qquad \mathcal{A} = \{A(a), B(c), R(a,b)\}, \qquad \mathcal{N} = \{B(b)\}.$$

Consider a model $\mathcal{I}_1 \in \mathit{Mod}(\mathcal{K})$ and models $\mathcal{J}_0, \mathcal{J}_1 \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$:

$$\mathcal{I}_1 = \{A(a), B(c), R(a,b)\}, \quad \mathcal{J}_0 = \{B(b), B(c)\}, \quad \mathcal{J}_1 = \{A(a), B(b), R(a,c)\}.$$

The fact that $\mathcal{J}_1 \in \mathcal{I}_1 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ can be shown in the same way as in Case A3. Now we will show that $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\#}$. Clearly, $\mathcal{J}_1 \notin \mathcal{I}_1 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\#}$. Indeed,

$$|\mathcal{I}_1 \ominus \mathcal{J}_0| = |\{A(a), B(b), R(a,b)\}| = 3 < 4 = |\mathcal{I}_1 \ominus \mathcal{J}_1| = |\{B(c), B(b), R(a,b), R(a,c)\}|.$$

Assume that there is $\mathcal{I}'_1 \models \mathcal{T} \cup \mathcal{A}$ different from $\mathcal{I}_1$ such that $\mathcal{J}_1 \in \mathcal{I}'_1 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\#}$. Due to Proposition 17, $\mathit{dist}^a_{\#} = |\mathcal{I}'_1 \ominus \mathcal{J}_1|$ is finite. It is easy to see that

$$\mathcal{I}'_1 \ominus \mathcal{J}_0 = (\mathcal{I}_1 \ominus \mathcal{J}_0) \cup \mathit{Remain}_0, \qquad \mathcal{I}'_1 \ominus \mathcal{J}_1 = (\mathcal{I}_1 \ominus \mathcal{J}_1) \cup \mathit{Remain}_1,$$

where $\mathit{Remain}_i = (\mathcal{I}'_1 \ominus \mathcal{J}_i) \setminus (\mathcal{I}_1 \ominus \mathcal{J}_i)$. Clearly, $|\mathcal{I}'_1 \ominus \mathcal{J}_0| = 3 + |\mathit{Remain}_0|$ and $|\mathcal{I}'_1 \ominus \mathcal{J}_1| = 4 + |\mathit{Remain}_1|$. We now show that $|\mathcal{I}'_1 \ominus \mathcal{J}_0| < |\mathcal{I}'_1 \ominus \mathcal{J}_1|$ which will give a contradiction. Due to finiteness of $|\mathcal{I}'_1 \ominus \mathcal{J}_1|$, it suffices to show that $\mathit{Remain}_0 = \mathit{Remain}_1$. Since $B(c)$ is in $\mathcal{J}_0$ and not in $\mathcal{I}'_1 \ominus \mathcal{J}_0$ and $A(a), R(a,c)$ are in $\mathcal{J}_1$ and not in $\mathcal{I}'_1 \ominus \mathcal{J}_1$, we need to check the following three cases: *(i)* $B(c) \notin \mathit{Remain}_1$, *(ii)* $A(a) \notin \mathit{Remain}_0$, and *(iii)* $R(a,c) \notin \mathit{Remain}_0$. Concerning Case (i), it holds by definition of Remain_1 since $B(c) \in \mathcal{I}_1 \ominus \mathcal{J}_1$. Case (ii) holds by the similar reason. Finally, Case (iii) holds since $R(a,c) \notin \mathcal{I}$ (note that $B(c) \in \mathcal{A}$ and $\exists R^- \sqsubseteq \neg B \in \mathcal{T}$) and $R(a,c) \notin \mathcal{J}_0$, and therefore $R(a,c) \notin \mathcal{I} \ominus \mathcal{J}_0$. Thus, $\mathit{Remain}_0 = \mathit{Remain}_1$ and we conclude the proof.

Case A6: $\mathcal{G}^a_{\subseteq} \not\leq_{sem} \mathcal{L}^a_{\#}$. Consider the following KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and a new information $\mathcal{N}$:

$$\mathcal{T} = \{A \sqsubseteq \exists R, \exists R^- \sqsubseteq \neg B, C \sqsubseteq \neg\exists R\}, \qquad \mathcal{A} = \{C(a), B(b)\}, \qquad \mathcal{N} = \{A(a)\}.$$

Consider a model $\mathcal{I}_1 \in \mathit{Mod}(\mathcal{K})$ and models $\mathcal{J}_0, \mathcal{J}_1 \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$:

$$\mathcal{I}_1 = \{C(a), B(b)\}, \quad \mathcal{J}_0 = \{A(a), R(a,d), B(b)\}, \quad \mathcal{J}_1 = \{A(a), R(a,b)\}.$$

Clearly, $\mathcal{J}_0 \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{J}_1 \models \mathcal{T} \cup \mathcal{N}$. We now prove that $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\subseteq}$, but not under $\mathcal{L}^a_{\#}$.

Assume that $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\subseteq}$. Then, there are $\mathcal{I}' \models \mathcal{K}$ and $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ such that

$$\mathcal{I}' \ominus \mathcal{J}' \subsetneq \mathcal{I}_1 \ominus \mathcal{J}_1 = \{A(a), C(a), R(a,b), B(b)\} \tag{A.3}$$

First observe that $\mathcal{I}' \ominus \mathcal{J}'$ contains two atoms $A(a)$ and $C(a)$ since $\mathcal{T} \models_{\mathcal{T}} A \sqsubseteq \neg C$, $\mathcal{A} \models_{\mathcal{T}} C(a)$, and $\mathcal{N} \models_{\mathcal{T}} A(a)$. Thus, one of the two cases is possible: *(i)* $R(a,b) \notin \mathcal{I}' \ominus \mathcal{J}'$ or *(ii)* $B(b) \notin \mathcal{I}' \ominus \mathcal{J}'$. Assume Case (i) holds. Due to $\mathcal{A} \models_{\mathcal{T}} C(a)$ and $\mathcal{T} \models A \sqsubseteq \neg C$, it holds that $R(a,v) \notin \mathcal{I}$ for every constant v, and in particular for $v = b$. Thus, $R(a,b) \notin \mathcal{J}'$. Since $\mathcal{N} \models_{\mathcal{T}} A(a)$, there is $R(a,v')$ in $\mathcal{J}'$ for some constant $v' \neq b$. Since $R(a,v') \notin \mathcal{I}'$, we conclude that $R(a,v') \in \mathcal{I}' \ominus \mathcal{J}'$ which contradicts Equation A.3. Assume Case (ii) holds. Due to $\mathcal{A} \models_{\mathcal{T}} B(b)$ we have that $B(b) \in \mathcal{I}'$. Since $B(b) \notin \mathcal{I}' \ominus \mathcal{J}'$, we conclude that $B(b) \in \mathcal{J}'$. As we discussed above, there is $R(a,v')$ in $\mathcal{J}'$ for some constant v'. Due to $\exists R^- \sqsubseteq \neg B$ and $B(b) \in \mathcal{J}'$, we conclude that $R(a,b) \notin \mathcal{J}'$. Therefore, there is $R(a,v) \in \mathcal{J}'$ for $v \neq b$. As we showed above $R(a,v) \notin \mathcal{I}'$, thus $R(a,v) \in \mathcal{I}' \ominus \mathcal{J}'$ which contradicts Equation A.3. Hence, we conclude that, $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\subseteq}$.

We now show that $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^a_\#$. First observe that $\mathcal{J}_1 \notin \mathcal{I}_1 \diamond \mathcal{N}$ under $\mathcal{L}^a_\#$. Indeed,

$$|\mathcal{I}_1 \ominus \mathcal{J}_0| = |\{A(a), C(a), R(a,d)\}| = 3 < 4 = |\mathcal{I}_1 \ominus \mathcal{J}_1| = |\{A(a), C(a), R(a,b), B(b)\}|.$$

Assume there is $\mathcal{I}'_1$ different from $\mathcal{I}_1$ such that $\mathcal{J}_1 \in \mathcal{I}'_1 \diamond \mathcal{N}$ under $\mathcal{L}^a_\#$. Due to Proposition 17, $\mathit{dist}^a_\#(\mathcal{J}_1, \mathcal{I}'_1) = |\mathcal{I}'_1 \ominus \mathcal{J}_1|$ is finite. From the discussion above it follows that:

$$\mathcal{I}'_1 \ominus \mathcal{J}_0 = (\mathcal{I}_1 \ominus \mathcal{J}_0) \cup \mathit{Remain}_0, \qquad \mathcal{I}'_1 \ominus \mathcal{J}_1 = (\mathcal{I}_1 \ominus \mathcal{J}_1) \cup \mathit{Remain}_1,$$

where $\mathit{Remain}_i = (\mathcal{I}'_1 \ominus \mathcal{J}_i) \setminus (\mathcal{I}_1 \ominus \mathcal{J}_i)$. Clearly, $|\mathcal{I}'_1 \ominus \mathcal{J}_0| = 3 + |\mathit{Remain}_0|$ and $|\mathcal{I}'_1 \ominus \mathcal{J}_1| = 4 + |\mathit{Remain}_1|$. As in Case A5, we now show that $|\mathcal{I}'_1 \ominus \mathcal{J}_0| < |\mathcal{I}'_1 \ominus \mathcal{J}_1|$ which will give a contradiction. Due to finiteness of $|\mathcal{I}'_1 \ominus \mathcal{J}_1|$, it suffices to show that $\mathit{Remain}_0 = \mathit{Remain}_1$. Note that $\mathcal{J}_0$ and $\mathcal{J}_1$ differ only on the following atoms: $R(a,d), B(d)$ are in $\mathcal{J}_0$ and not in $\mathcal{J}_1$, and $R(a,b)$ is in $\mathcal{J}_1$ and not in $\mathcal{J}_0$. Since $R(a,d), B(d)$ are in $\mathcal{I}'_1 \ominus \mathcal{J}_0$ and $R(a,b)$ is $\mathcal{I}'_1 \ominus \mathcal{J}_1$, we need to check is the following three cases: *(i)* $R(a,d) \notin \mathit{Remain}_1$, *(ii)* $B(d) \notin \mathit{Remain}_1$, and *(iii)* $R(a,b) \notin \mathit{Remain}_0$. Concerning Case (i), it holds since $R(a,d) \notin \mathcal{I}'_1$ and $R(a,d) \notin \mathcal{J}_1$. The same argument applies for Case (iii). Case (ii) holds by definition of Remain_1 since $B(d) \in \mathcal{I}_1 \ominus \mathcal{J}_1$. Thus, $\mathit{Remain}_0 = \mathit{Remain}_1$ and we conclude the proof.

Case A7: $\mathcal{G}^a_\subseteq \not\preccurlyeq_{sem} \mathcal{G}^a_\#$. Assume that $\mathcal{G}^a_\subseteq \preccurlyeq_{sem} \mathcal{G}^a_\#$. Then, due to the fact that $\mathcal{G}^a_\# \preccurlyeq_{sem} \mathcal{L}^a_\#$ (see Theorem 6), and transitivity of $\preccurlyeq_{sem}$-relation, we conclude that $\mathcal{G}^a_\subseteq \preccurlyeq_{sem} \mathcal{L}^a_\#$, which gives a contradiction with Case A6.

Case A8: $\mathcal{L}^a_\# \not\preccurlyeq_{sem} \mathcal{L}^a_\subseteq$. Consider the following KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and the new information $\mathcal{N}$:

$$\mathcal{T} = \{A \sqsubseteq \neg\exists R\}, \qquad \mathcal{A} = \{A(c), R(a,b)\}, \qquad \mathcal{N} = \{A(a), R(c,d)\}.$$

Consider the following models $\mathcal{I}_0 \in \mathit{Mod}(\mathcal{K})$ and $\{\mathcal{J}_0, \mathcal{J}_1\} \subseteq \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$:

$$\mathcal{I}_0 = \{A(c), R(a,b)\} \cup \{R(a,\alpha_i)\}_{i=1}^{\infty}, \quad \mathcal{J}_0 = \{A(a), R(c,d)\}, \quad \mathcal{J}_1 = \{A(a), R(c,d)\} \cup \{R(c,\beta_j)\}_{j=1}^{\infty},$$

where $\alpha_i, \beta_j \in \mathit{elta} \setminus \mathit{adom}(\mathcal{K})$ for $i, j \geqslant 1$. Consider also two sets of atoms:

$$\mathcal{S}_0 = \{A(a), A(c), R(a,b), R(c,d)\} \cup \{R(a,\alpha_i)\}_{i=1}^{\infty}$$
$$\mathcal{S}_1 = \{A(a), A(c), R(a,b), R(c,d)\} \cup \{R(c,\beta_j)\}_{j=1}^{\infty}.$$

Clearly, for every $\mathcal{J} \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$ it holds that $\mathcal{S}_0 \subseteq \mathcal{I}_0 \ominus \mathcal{J}$. Since $\mathcal{I}_0 \ominus \mathcal{J}_0 = \mathcal{S}_0$, we conclude that $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$ under both $\mathcal{L}^a_\subseteq$ and $\mathcal{L}^a_\#$. Observe that $\mathcal{I}_0 \ominus \mathcal{J}_1 = \mathcal{S}_0 \cup \{R(c,\beta_j)\}_{j=1}^{\infty} = (\mathcal{I}_0 \ominus \mathcal{J}_0) \cup \{R(c,\beta_j)\}_{j=1}^{\infty}$. Since $|\mathcal{I}_0 \ominus \mathcal{J}_0| = \omega$, we obtain that $|\mathcal{I}_0 \ominus \mathcal{J}_1| = \omega$ and therefore $\mathit{dist}^a_\#(\mathcal{I}_0, \mathcal{J}_0) = \mathit{dist}^a_\#(\mathcal{I}_0, \mathcal{J}_1)$ holds, which gives that $\mathcal{J}_1 \in \mathcal{I}_0 \diamond \mathcal{N}$ under $\mathcal{L}^a_\#$. We now show that $\mathcal{J}_1 \notin \mathcal{I}_0 \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$, and thus obtain a contradiction. Let $\mathcal{I}$ be a model of $\mathcal{K}$ and $\mathcal{S} = (\mathcal{I} \ominus \mathcal{J}_1) \setminus \mathcal{S}_1$. Clearly, $\mathcal{I} \ominus \mathcal{J}_1 = \mathcal{S}_1 \cup \mathcal{S}$. Now observe that $\mathcal{I} \ominus \mathcal{J}_0 = \mathcal{S}_1 \cup \mathcal{S} \setminus \{R(c,\beta_j)\}_{j=1}^{\infty}$, since $\mathcal{J}_0$ and $\mathcal{J}_1$ differ only on the set $\{R(c,\beta_j)\}_{j=1}^{\infty}$, which is disjoint from $\mathcal{I} \ominus \mathcal{J}_0$ since it belongs to neither $\mathcal{I}$ nor $\mathcal{J}_0$. Thus, $\mathcal{J}_1 \notin \mathcal{I}_0 \diamond \mathcal{N}$ under $\mathcal{L}^a_\subseteq$, which concludes the proof.

We continue with semantics which are based on symbols. Note that in order to improve readability of the proof, the cases Ai and Si are symmetric in the sense that for each *i* the semantics compared in Ai and Si are different in the upper-indexes only, i.e., if in Ai we compare two atom-based semantics, then in Si we compare the corresponding symbol based ones. Since in Case A8 we have $\mathcal{L}^a_\# \not\preccurlyeq_{sem} \mathcal{L}^a_\subseteq$, while $\mathcal{L}^s_\# \preccurlyeq_{sem} \mathcal{L}^s_\subseteq$ holds, there is no Case S8.

Case S1: $\mathcal{L}^s_\# \not\preccurlyeq_{sem} \mathcal{G}^s_\subseteq$. This case holds already for *DL-Lite*$^{pr}_{\mathcal{R}}$, which is a sub-language of *DL-Lite*$_{\mathcal{FR}}$ (shown in Theorem 20), consequently it holds for *DL-Lite*$_{\mathcal{FR}}$.

Case S2: $\mathcal{L}^s_{\#} \not\preccurlyeq_{sem} \mathcal{G}^s_{\#}$. Assume that $\mathcal{L}^s_{\#} \preccurlyeq_{sem} \mathcal{G}^s_{\#}$. Then, due to the fact that $\mathcal{G}^s_{\#} \preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$ (see Theorem 6) and transitivity of $\preccurlyeq_{sem}$-relation, we conclude that $\mathcal{L}^s_{\#} \preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$, which contradicts Case S1.

Case S3: $\mathcal{L}^s_{\subseteq} \not\preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$. Assume that $\mathcal{L}^s_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$. Then, due to the fact that for *DL-Lite*$_{\mathcal{FR}}$ $\mathcal{L}^s_{\#} \preccurlyeq_{sem} \mathcal{L}^s_{\subseteq}$ (see Theorem 6) and transitivity of $\preccurlyeq_{sem}$-relation, we conclude that $\mathcal{L}^s_{\#} \preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$, which contradicts Case S1.

Case S4: $\mathcal{L}^s_{\subseteq} \not\preccurlyeq_{sem} \mathcal{G}^s_{\#}$. Assume that $\mathcal{L}^s_{\subseteq} \preccurlyeq_{sem} \mathcal{G}^s_{\#}$. Then, due to the fact that for *DL-Lite*$_{\mathcal{FR}}$ $\mathcal{L}^s_{\#} \preccurlyeq_{sem} \mathcal{L}^s_{\subseteq}$ and $\mathcal{G}^s_{\#} \preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$ (see Theorem 6), and transitivity of $\preccurlyeq_{sem}$-relation, we conclude that $\mathcal{L}^s_{\#} \preccurlyeq_{sem} \mathcal{G}^s_{\subseteq}$, which contradicts Case S1.

Case S5: $\mathcal{L}^s_{\subseteq} \not\preccurlyeq_{sem} \mathcal{L}^s_{\#}$. Consider the following KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and a new information $\mathcal{N}$:

$$\mathcal{T} = \{A \sqsubseteq \exists R,\ \exists R^- \sqsubseteq D,\ \exists R \sqsubseteq \neg C,\ \exists R^- \sqsubseteq \neg B_1,\ \exists R^- \sqsubseteq \neg B_2,\ B_1 \sqsubseteq \neg E,\ B_2 \sqsubseteq \neg E\},$$

$$\mathcal{A} = \{C(a),\ B_1(b),\ B_2(b),\ D(b),\ E(d)\}, \qquad \mathcal{N} = \{A(a)\}.$$

Consider models $\mathcal{I}_1 \in Mod(\mathcal{K})$ and $\mathcal{J}_1 \in Mod(\mathcal{T} \cup \mathcal{N})$:

$$\mathcal{I}_1 = \{C(a),\ B_1(b),\ B_2(b),\ D(b),\ E(d)\}, \quad \mathcal{J}_1 = \{A(a),\ R(a,b),\ D(b),\ E(d)\}.$$

We now prove that $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^s_{\subseteq}$ and $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^s_{\#}$.

Assume that $\mathcal{J}_1 \notin \mathcal{I}_1 \diamond \mathcal{N}$ under $\mathcal{L}^s_{\subseteq}$, that is, there exists a model $\mathcal{J}_1' \in Mod(\mathcal{T} \cup \mathcal{N})$ such that

$$dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1') \subsetneq dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1) = \{A, C, R, B_1, B_2\}. \tag{A.4}$$

Observe that $A, C, R \in dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1')$ since $A \sqsubseteq \exists R$ and $\exists R \sqsubseteq \neg C$ are in $\mathcal{T}$, $C(a)$ is in $\mathcal{A}$, and $A(a)$ is in $\mathcal{N}$. Then, B_1 or B_2 is not in $dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1')$. Assume that $B_1 \notin dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1')$ (the case of B_2 is analogous). Therefore, it holds that $B_1^{\mathcal{I}_1} = B_1^{\mathcal{J}_1'}$, i.e., $B_1(b) \in \mathcal{J}_1'$, and due to $\exists R^- \sqsubseteq \neg B_1$ and $\mathcal{J}_1' \models \mathcal{T}$, we conclude that $R(a,b) \notin \mathcal{J}_1'$. Since $A(a) \in \mathcal{J}_1'$ and $A \sqsubseteq \exists R, \exists R \sqsubseteq D \in \mathcal{T}$, there exists an element $x \in \Delta$ such that $R(a,x), D(x) \in \mathcal{J}_1'$ and $x \neq b$. Hence, $D^{\mathcal{I}_1} \neq D^{\mathcal{J}_1'}$ and consequently $D \in dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1')$, which contradicts Equality (A.4). The contradiction yields that $B_1 \in dist^s_{\subseteq}(\mathcal{I}_1, \mathcal{J}_1')$. Thus, $\mathcal{J}_1 \in \mathcal{I}_1 \diamond \mathcal{N}$ under $\mathcal{L}^s_{\subseteq}$ and therefore $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^s_{\subseteq}$.

Now we show that $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^s_{\#}$. Consider an arbitrary model $\mathcal{I} \in Mod(\mathcal{K})$. Since *(i)* $C(a), B_1(b), B_2(b)$ are in $\mathcal{I}$ while not in $\mathcal{J}_1$, and *(ii)* $A(a), R(a,b)$ are in $\mathcal{J}_1$ while not in $\mathcal{I}$, it holds that $dist^s_{\#}(\mathcal{I}, \mathcal{J}_1) \geq |\{A, C, R, B_1, B_2\}| = 5$. Consider the following model:

$$\mathcal{J}' = (\mathcal{I} \setminus \{C(a)\}) \cup \{A(a), R(a,d), D(d)\},$$

It is easy to see that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ due to the construction of $\mathcal{J}'$. Indeed, $\mathcal{J}' \models \mathcal{N}$ since $A(a) \in \mathcal{J}'$. To show that $\mathcal{J}' \models \mathcal{T}$ we need to show that all the assertions in $\mathcal{T}$ are satisfied by $\mathcal{J}'$:

- $\mathcal{J}' \models A \sqsubseteq \exists R$ iff for every $x \in A^{\mathcal{J}'}$ there exists $y \in \Delta$ such that $(x,y) \in R^{\mathcal{J}'}$. Note that $A^{\mathcal{J}'} = A^{\mathcal{I}} \cup \{a\}$. If $x \in A^{\mathcal{I}}$, then the requirement holds since $\mathcal{I} \models A \sqsubseteq \exists R$ and $R^{\mathcal{I}} \subseteq R^{\mathcal{J}'}$. If $x = a$, then it holds since $(a,d) \in R^{\mathcal{J}'}$.
- $\mathcal{J}' \models \exists R^- \sqsubseteq D$ iff for every y such that $(x,y) \in R^{\mathcal{J}'}$ for some $x \in \Delta$, it holds that $y \in D^{\mathcal{J}'}$. Similarly to the previous case, in the case when $(x,y) \in R^{\mathcal{I}}$ it holds since $\mathcal{I} \models \exists R^- \sqsubseteq D$ and $D^{\mathcal{I}} \subseteq D^{\mathcal{J}'}$. In the case when $y = d$, it holds since $d \in D^{\mathcal{J}'}$ by construction.
- $\mathcal{J}' \models \exists R \sqsubseteq \neg C$ iff for every $x \in C^{\mathcal{J}'}$ there is no $y \in \Delta$ such that $(x,y) \in R^{\mathcal{J}'}$. This requirement holds since $C^{\mathcal{J}'} = C^{\mathcal{I}} \setminus \{C(a)\}$ and $R^{\mathcal{J}'} = R^{\mathcal{I}} \cup \{R(a,d)\}$ and $\mathcal{I} \models \exists R \sqsubseteq \neg C$.
- $\mathcal{J}' \models \exists R^- \sqsubseteq \neg B_i$ iff for every $y \in B_i^{\mathcal{J}'}$ there is no $x \in \Delta$ such that $(x,y) \in R^{\mathcal{J}'}$, where $i \in \{1,2\}$. This requirement holds since $R^{\mathcal{J}'} = R^{\mathcal{I}} \cup \{R(a,d)\}$, $B_i^{\mathcal{J}'} = B_i^{\mathcal{I}}$, and $d \notin B_i^{\mathcal{I}}$ for $i \in \{1,2\}$.

- Finally, $\mathcal{J}' \models B_i \sqsubseteq \neg E$ iff for every $x \in B_i^{\mathcal{I}}$ it holds that $x \notin E^{\mathcal{I}}$, where $i \in \{1,2\}$. This requirement holds since $\mathcal{I} \models B_i \sqsubseteq \neg E$ and $B_i^{\mathcal{J}'} = B_i^{\mathcal{I}}$ and $E^{\mathcal{J}'} = E^{\mathcal{I}}$.

It is easy to check that $\mathit{dist}^s_{\#}(\mathcal{I}, \mathcal{J}') = |\{A, C, R, D\}| = 4$. To sum up, for every model $\mathcal{I} \in \mathit{Mod}(\mathcal{K})$ there exists a model $\mathcal{J}' \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$ such that $\mathit{dist}^s_{\#}(\mathcal{I}_1, \mathcal{J}_1') \lneq \mathit{dist}^s_{\#}(\mathcal{I}_1, \mathcal{J}_1)$. That is, $\mathcal{J}' \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^s_{\#}$, which concludes the proof.

Case S6: $\mathcal{G}^s_{\subseteq} \not\preccurlyeq_{\mathit{sem}} \mathcal{L}^s_{\#}$. Analogous to Case A6: one can mimic the same symmetric difference as in A6 by insuring that for each atom in the symmetric differences we have a corresponding new symbol.

Case S7: $\mathcal{G}^s_{\subseteq} \not\preccurlyeq_{\mathit{sem}} \mathcal{G}^s_{\#}$. Assume that $\mathcal{G}^s_{\subseteq} \preccurlyeq_{\mathit{sem}} \mathcal{G}^s_{\#}$. Then, due to the fact that $\mathcal{G}^s_{\#} \preccurlyeq_{\mathit{sem}} \mathcal{L}^s_{\#}$ (see Theorem 6), and transitivity of $\preccurlyeq_{\mathit{sem}}$-relation, we conclude that $\mathcal{G}^s_{\subseteq} \preccurlyeq_{\mathit{sem}} \mathcal{L}^s_{\#}$, which contradicts Case S6.

Finally, we will show that $\mathcal{S}_1 \not\preccurlyeq_{\mathit{sem}} \mathcal{S}_2$ whenever one of $\{\mathcal{S}_1, \mathcal{S}_2\}$ is an atom-based and the other one is a symbols-based semantics. Due to Theorem6 and transitivity of $\preccurlyeq_{\mathit{sem}}$-relation, the case when $\mathcal{S}_1$ is a symbol-based semantics and $\mathcal{S}_2$ is an atom-based one requires to prove two relationships: $\mathcal{G}^s_{\#} \not\preccurlyeq_{\mathit{sem}} \mathcal{L}^a_{\#}$ and $\mathcal{G}^s_{\#} \not\preccurlyeq_{\mathit{sem}} \mathcal{L}^a_{\subseteq}$. The case when $\mathcal{S}_1$ is an atom-based semantics and $\mathcal{S}_2$ is a symbol-based one is slightly easier since for *DL-Lite*$_{\mathcal{FR}}$ the relationship $\mathcal{L}^s_{\#} \preccurlyeq_{\mathit{sem}} \mathcal{L}^s_{\subseteq}$ holds. This case requires to prove only one relationship: $\mathcal{G}^a_{\#} \not\preccurlyeq_{\mathit{sem}} \mathcal{L}^s_{\subseteq}$. In particular, this follows from the fact that $\mathcal{G}^a_{\#}$ is the least elements among the atom-based semantics, and $\mathcal{L}^s_{\subseteq}$ is the greatest one among the symbol-based semantics.

Case AS: $\mathcal{G}^a_{\#} \not\preccurlyeq_{\mathit{sem}} \mathcal{L}^s_{\subseteq}$. Consider the following KB $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ and a new information $\mathcal{N}$:

$$\begin{aligned}\mathcal{T} = \{&B \sqsubseteq \exists Q,\ \exists Q \sqsubseteq B, \exists Q^- \sqsubseteq \exists R,\ \exists R \sqsubseteq \exists Q^-,\ A \sqsubseteq \exists R^-,\ \exists R^- \sqsubseteq A,\ A \sqsubseteq \neg C,\\ &(\mathsf{func}\ R),\ (\mathsf{func}\ R^-),\ (\mathsf{func}\ Q),\ (\mathsf{func}\ Q^-)\},\\ &\mathcal{A} = \{Q(a,b),\ R(b,c),\ C(d)\}, \qquad \mathcal{N} = \{C(c)\}.\end{aligned}$$

Consider models $\mathcal{I}_1 \in \mathit{Mod}(\mathcal{K})$ and $\mathcal{J}_1 \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$:

$$\mathcal{I}_1 = \{B(a),\ Q(a,b),\ R(b,c),\ A(c),\ C(d)\}, \qquad \mathcal{J}_1 = \{C(c), C(d)\}.$$

We now prove that $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\#}$ and $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^s_{\subseteq}$.

Suppose that $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\#}$. In particular, this means that there exist models $\mathcal{I}' \in \mathit{Mod}(\mathcal{K})$ and $\mathcal{J}' \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$ such that

$$d' = |\mathcal{I}' \ominus \mathcal{J}'| \lneq |\mathcal{I}_1 \ominus \mathcal{J}_1| = |\{C(a), A(c), R(b,c), Q(a,b), B(a)\}| = 5. \tag{A.5}$$

It is easy to see that $C(a), A(c), R(b,c) \subseteq \mathcal{I}' \ominus \mathcal{J}'$, and therefore d' is equal to either 3 or 4. If $d' = 3$, then $Q(a,b) \notin \mathcal{I}' \ominus \mathcal{J}'$, and, since $Q(a,b) \in \mathcal{I}'$, we conclude that $Q(a,b) \in \mathcal{J}'$. Since $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$, it holds that there is an element $x \in \Delta$ such that $x \neq c$ and $R(b,x) \in \mathcal{J}'$. Due to the functionality of R and the fact that $R(b,c) \in \mathcal{I}'$, it holds that $R(b,x) \notin \mathcal{I}'$, thus $R(b,x) \in \mathcal{I}' \ominus \mathcal{J}'$, which contradicts the assumption that $d' = 3$. If $d' = 4$, then there are the following options:

- $Q(a,b) \notin \mathcal{J}'$, and consequently $\mathcal{I}' \ominus \mathcal{J}' = \{C(c), A(a), R(b,c), Q(a,b)\}$. Since $B(a) \in \mathcal{I}'$, we conclude that $B(a) \in \mathcal{J}'$. The last fact along with $\mathcal{J}' \models \mathcal{T}$ yields that there is an element $x \in \Delta$ such that $x \neq b$ and $Q(a,x) \in \mathcal{J}'$. Due to the functionality of Q and the fact that $Q(a,b) \in \mathcal{I}'$, it holds that $Q(a,x) \notin \mathcal{I}'$, thus $Q(a,x) \in \mathcal{I}' \ominus \mathcal{J}'$, which contradicts the assumption that $d' = 4$.
- $Q(a,b) \in \mathcal{J}'$, and consequently there is an element $x \in \Delta$ such that $x \neq c$ and $R(b,x) \in \mathcal{J}'$. Due to the functionality of R and the fact that $R(b,c) \in \mathcal{I}'$, it holds that $R(b,x) \notin \mathcal{I}'$, thus $R(b,x) \in \mathcal{I}' \ominus \mathcal{J}'$. So, $\mathcal{I}' \ominus \mathcal{J}' = \{C(c), A(c), R(b,c), R(b,x)\}$. Then, since $\mathcal{J}' \models \mathcal{T}$, it holds then $A(x) \in \mathcal{J}'$ and consequently $A(x) \in \mathcal{I}'$. The latter membership yields, due to $A \sqsubseteq \exists R^- \in \mathcal{T}$, that there is an element $y \in \Delta$ such that $y \neq b$ and $R(y,x) \in \mathcal{I}'$. Since $R(y,x) \notin \mathcal{I}' \ominus \mathcal{J}'$, we conclude that $R(y,x) \in \mathcal{J}'$. Hence, we have that both $R(b,x)$ and $R(y,x)$ are in $\mathcal{J}'$, which contradicts the functionality of R^-.

Thus, there are no models $\mathcal{I}'$ and $\mathcal{J}'$ as in Equation (A.5), and therefore $\mathcal{J}_1 \in \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{G}^a_{\#}$.

We show now that $\mathcal{J}_1 \notin \mathcal{K} \diamond \mathcal{N}$ under $\mathcal{L}^s_{\subseteq}$, that is, for every model $\mathcal{I} \in \mathit{Mod}(\mathcal{K})$ there is a model $\mathcal{J} \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$ such that

$$\mathit{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}) \subsetneq \mathit{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}_1). \tag{A.6}$$

Suppose that Equality (A.6) does not hold and there is a model $\mathcal{I}'' \in \mathit{Mod}(\mathcal{K})$ such that $\mathcal{J}_1 \in \mathcal{I}'' \diamond \mathcal{N}$ under $\mathcal{L}^s_{\subseteq}$. Since $\mathcal{I}'' \in \mathit{Mod}(\mathcal{K})$ and $\mathcal{J}_1 \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$, it holds that $B(a), Q(a,b), R(b,c), A(c) \in \mathcal{I}$ and $C(c) \notin \mathcal{I}$, while $B(a), Q(a,b), R(b,c), A(c) \notin \mathcal{J}_1$ and $C(c) \in \mathcal{J}_1$. So, we conclude that $\{A, B, C, Q, R\} \subseteq \mathit{dist}^s_{\subseteq}(\mathcal{I}, \mathcal{J}_1)$. Consider a model $\mathcal{J}''$ as follows:

$$\mathcal{J}'' = (\mathcal{I}'' \setminus \{A(c), R(b,c), C(d)\}) \cup \{R(b,d), A(d), C(c)\}.$$

It is easy to see that

$$\mathit{dist}^s_{\subseteq}(\mathcal{I}'', \mathcal{J}'') = \{A, C, R\} \subsetneq \{A, B, C, Q, R\}.$$

It remains to prove that $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ since it will contradict the fact that $\mathcal{J}_1 \in \mathcal{I}'' \diamond \mathcal{N}$ under $\mathcal{L}^s_{\subseteq}$. It is easy to see that $\mathcal{J}'' \models \mathcal{N}$ since $C(c) \in \mathcal{J}''$ by construction. We prove now that $\mathcal{J}''$ also satisfies $\mathcal{T}$. We have the following cases:

- Assertions $B \sqsubseteq \exists Q$, $\exists Q \sqsubseteq B$, $(\mathsf{func}\ Q)$, and $(\mathsf{func}\ Q^-)$ are satisfied trivially due to the fact that $\mathcal{I}''$ satisfies them and $B^{\mathcal{I}''} = B^{\mathcal{J}''}$ and $Q^{\mathcal{I}''} = Q^{\mathcal{J}''}$.
- $\mathcal{J}''$ satisfies $\alpha = \exists Q^- \sqsubseteq \exists R$ iff for every $x \in (\exists Q^-)^{\mathcal{I}''}$ there exists $y \in \Delta$ such that $(x, y) \in R^{\mathcal{J}''}$. Indeed, if $x = b$, then $\mathcal{J}'' \models \alpha$ since $R(b,d) \in \mathcal{J}''$ by construction. In the case when $x \in \Delta \setminus \{b\}$, the requirement holds since $(\exists Q^-)^{\mathcal{J}''} = (\exists Q^-)^{\mathcal{I}''}$ and $R^{\mathcal{J}''} = (R^{\mathcal{I}''} \setminus \{(b,c)\}) \cup \{(b,d)\}$, and therefore the existence of such y in $(\exists R^-)^{\mathcal{J}''}$, follows from its existence in $(\exists R^-)^{\mathcal{I}''}$.
- $\mathcal{J}''$ satisfies $\exists R \sqsubseteq \exists Q^-$ since $\mathcal{I}''$ does so and $(\exists R)^{\mathcal{I}''} = (\exists R)^{\mathcal{J}''}$ and $(\exists Q^-)^{\mathcal{I}''} = (\exists Q^-)^{\mathcal{J}''}$.
- $\mathcal{J}''$ satisfies $A \sqsubseteq \exists R^-$ and $\exists R^- \sqsubseteq A$ since $\mathcal{I}''$ does so and $A^{\mathcal{J}''} = (A^{\mathcal{I}''} \setminus \{c\}) \cup \{d\}$ and $(\exists R^-)^{\mathcal{J}''} = ((\exists R^-)^{\mathcal{I}''} \setminus \{c\}) \cup \{d\}$.
- $\mathcal{J}''$ satisfies $\alpha = A \sqsubseteq \neg C$ iff for every $x \in A^{\mathcal{J}''}$ it holds that $x \notin C^{\mathcal{J}''}$. The requirement is satisfied due to the fact that $\mathcal{I}'' \models \alpha$ and $A^{\mathcal{J}''} = (A^{\mathcal{I}''} \setminus \{c\}) \cup \{d\}$ and $C^{\mathcal{J}''} = (C^{\mathcal{I}''} \setminus \{d\}) \cup \{c\}$.
- $\mathcal{J}''$ satisfies $\alpha = (\mathsf{func}\ R)$ iff $(x, y) \in R^{\mathcal{J}''}$, there is *no* $z \in \Delta$ such that $z \neq y$ and $(x, z) \in R^{\mathcal{J}''}$. If $x = b$, then the requirement holds since *(i)* $R^{\mathcal{J}''} = (R^{\mathcal{I}''} \setminus \{(b,c)\}) \cup \{(b,d)\}$, *(ii)* $\mathcal{I}'' \models \mathcal{T}$ and therefore it does not hold that $(b, y) \in R^{\mathcal{I}''}$, where $y \neq c$. If $x \in \Delta \setminus \{b\}$, then the requirement holds due to Item (i) above.
- Finally, $\mathcal{J}''$ satisfies $(\mathsf{func}\ R^-)$ iff for every $y \in (\exists R^-)^{\mathcal{J}''}$ there is the only one $x \in \Delta$ such that $(x, y) \in R^{\mathcal{J}''}$. The requirement holds since $(\exists R^-)^{\mathcal{J}''} = ((\exists R^-)^{\mathcal{I}''} \setminus \{c\}) \cup \{d\}$ and there is no x such that $(x, d) \in R^{\mathcal{I}''}$ since $C(d) \in \mathcal{I}''$ and $\mathcal{T} \models \exists R^- \sqsubseteq \neg C$.

Thus, $\mathcal{J}'' \in \mathit{Mod}(\mathcal{T} \cup \mathcal{N})$, which concludes the proof.

Case SA1: $\mathcal{G}^s_{\#} \not\leq_{\mathit{sem}} \mathcal{L}^a_{\#}$. This case holds already for *DL-Lite*$^{pr}_{\mathcal{R}}$, which is a sub-language of *DL-Lite*$_{\mathcal{FR}}$ (shown in Theorem 20), consequently it holds for *DL-Lite*$_{\mathcal{FR}}$.

Case SA2: $\mathcal{G}^s_{\#} \not\leq_{\mathit{sem}} \mathcal{L}^a_{\subseteq}$. This case holds already for *DL-Lite*$^{pr}_{\mathcal{R}}$, which is a sub-language of *DL-Lite*$_{\mathcal{FR}}$ (shown in Theorem 20), consequently it holds for *DL-Lite*$_{\mathcal{FR}}$.

□

Appendix B. Proofs for Section 4.1

We abbreviate the term negative inclusion assertion to *NI* and positive inclusion assertion to *PI*. In this and the following section we will need the following property of *DL-Lite*$_{\mathcal{R}}$.

Proposition 49. *Let $\mathcal{T} \cup \mathcal{A}$ be a satisfiable DL-Lite$_\mathcal{R}$ KB and L be a membership assertion. If $\mathcal{A} \models_\mathcal{T} L$, then there exists a membership assertion $L_0 \in \mathcal{A}$ such that $L_0 \models_\mathcal{T} L$.*

Proof. Assume that L is a positive assertion, i.e., of the form $P(a,b)$, $\exists R(c)$, or $A(c)$. Since *chase*$_\mathcal{T}(\mathcal{A})$ is a model of $\mathcal{T} \cup \mathcal{A}$, the entailment $\mathcal{A} \models_\mathcal{T} L$ implies that *chase*$_\mathcal{T}(\mathcal{A})$ models L. Thus, either $L \in \mathcal{A}$ when $L \in \{P(a,b), A(c)\}$ or $R(c,x) \in \mathcal{A}$ when $L = \exists R(c)$ and $x \notin$ *adom*$(\mathcal{A})$. By the definition of chase, for every atom in *chase*$_\mathcal{T}(\mathcal{A})$, there is a sequence of atoms $f_1, \ldots, f_n$, where *(i)* $f_n = L$ or $f_n = R(c,x)$, depending on the shape of L; *(ii)* $f_1 \in \mathcal{A}$, or $\exists R'(c') \in \mathcal{A}$ and $f_1 = R'(c', x')$, where $x' \notin$ *adom*$(\mathcal{A})$; *(iii)* each f_{i+1} is derivable from f_i by triggering a positive inclusion assertion of $\mathcal{T}$, that is, $f_i \models_\mathcal{T} f_{i+1}$. Due to transitivity of $\models_\mathcal{T}$, due to $f_1 \models_\mathcal{T} f_n$, and by taking $L_0 = f_1$ or $L_0 = \exists R'(c')$ depending on the shape of f_1, we obtain $L_0 \models_\mathcal{T} L$ and conclude the proof.

Assume that L is a negative inclusion assertion, i.e., $L = \neg A(c)$. If $L \in \mathcal{A}$, then taking $L_0 = L$ concludes the proof. Assume that $L \notin \mathcal{A}$. Assume that

$$\text{for every assertion } L' \in \mathcal{A} \text{ it holds } L' \not\models_\mathcal{T} L. \tag{B.1}$$

Let $L_1, \ldots, L_n$ be all the PIs of $\mathcal{A}$. Consider the interpretation:

$$\mathcal{I} = \bigcup_{i=1}^{n} \textit{chase}_\mathcal{T}(L_i) \cup \textit{chase}_\mathcal{T}(A(c)).$$

Clearly, $\mathcal{I} \not\models L$. We now show that $\mathcal{I} \models \mathcal{A} \cup \mathcal{T}$, thus we will obtain a contradiction with $\mathcal{A} \models_\mathcal{T} L$. Observe that $\mathcal{I}$ is a model of $\mathcal{A}$. Indeed, it models all the positive MAs of $\mathcal{A}$ by construction. Each *chase*$_\mathcal{T}(L_i)$ (and consequently their union) satisfies all negative MAs of $\mathcal{A}$. Assume there is i for which *chase*$_\mathcal{T}(L_i)$ does not satisfy an negative MA $\neg g$ of $\mathcal{A}$. Thus, $\{L_i, \neg g\} \models_\mathcal{T} \bot$ which contradicts satisfiability of $\mathcal{T} \cup \mathcal{A}$. Finally, *chase*$_\mathcal{T}(A(c))$ satisfies all negative MAs of $\mathcal{A}$. Assume it is not the case, and there is a negative MA $\neg g \in \mathcal{A}$ such that *chase*$_\mathcal{T}(A(c)) \models g$. Then, $A(c) \models_\mathcal{T} g$, and $\neg g \models_\mathcal{T} \neg A(c)$, thus we found an assertion in $\mathcal{A}$ that $\mathcal{T}$-entails L, which contradicts Equation B.1. Clearly, $\mathcal{I}$ models all the PIs of $\mathcal{T}$. It remains to show that $\mathcal{I}$ models each NI of $\mathcal{T}$. Assume there is a NI α such that $\mathcal{I} \not\models \alpha$. Then, there are two atoms f and f' in $\mathcal{I}$ such that $f \to \neg f'$ is an instantiation of the first-order interpretation of α. This implies that $\{f, \alpha\} \models \neg f'$. Clearly, ABoxes $\{L_i\}$ for $1 \leq i \leq n$ and $\{A(c)\}$ satisfy α, so does *chase*$_\mathcal{T}(L_i)$ for $1 \leq i \leq n$ and *chase*$_\mathcal{T}(A(c))$ due to Lemma 12 of [5]. This implies that $\{f, f'\} \not\subseteq$ *chase*$_\mathcal{T}(L_i)$ for each $1 \leq i \leq n$ and $\{f, f'\} \not\subseteq$ *chase*$_\mathcal{T}(A(c))$. Thus, two cases are possible:

(i) $f \in$ *chase*$_\mathcal{T}(L_i)$ for some $i \in \{1, \ldots, n\}$ and $f' \in$ *chase*$_\mathcal{T}(A(c))$ (the case when $f' \in$ *chase*$_\mathcal{T}(L)$ and $f \in$ *chase*$_\mathcal{T}(A(c))$ is symmetric),

(ii) $f \in$ *chase*$_\mathcal{T}(L_i)$ and $f' \in$ *chase*$_\mathcal{T}(L_j)$ for some different $i, j \in \{1, \ldots, n\}$.

In Case (i), $f' \in$ *chase*$(A(c))$ implies that $A(c) \models_\mathcal{T} f'$, and consequently $\neg f' \models_\mathcal{T} \neg A(c)$. Combining the latter entailment with $\{f, \alpha\} \models \neg f'$ we obtain $f \models_\mathcal{T} \neg A(c)$. Since $L_i \models_\mathcal{T} f$, we conclude that $L_i \models_\mathcal{T} \neg A(c)$ which contradicts the assumption in Equation B.1 and concludes the proof. In Case (ii), analogously to Case (i), we conclude that $L_i \models_\mathcal{T} \neg L_j$, thus A does not satisfy α which yields a contradiction with satisfiability of $\mathcal{T} \cup \mathcal{A}$. □

Proof of Proposition 14.
It follows from the definition of `AlignAlg` and the facts that in *DL-Lite*$_\mathcal{R}^{pr}$ disjointness that involve roles or their projections is forbidden and $\mathcal{N}$ contains only positive membership assertions. Indeed, let $\mathcal{A} \models_\mathcal{T} R(a,b)$ and `AlignAlg`$(\mathcal{K}, \mathcal{N}) \cup \mathcal{N} \not\models_\mathcal{T} R(a,b)$. Then, $\{R(a,b)\} \cup \mathcal{N} \models_\mathcal{T} \bot$ (see Algorithm 1). Thus, there are membership assertions L_1 and L_2, and a NI $\alpha \in cl(\mathcal{T})$ such that $\{R(a,b)\} \cup \mathcal{N} \models_\mathcal{T} \{L_1, L_2\}$ and

$\alpha \models L_1 \to \neg L_2$. Note that $L_1 \to \neg L_2$ should be seen as a first order formula with two subformulas L_1 and L_2, both without free variables. The semantics of this formula is defined straightforwardly: $\mathcal{I} \models L_1 \to \neg L_2$ if $\mathcal{I} \models L_1$ and $\mathcal{I} \not\models L_2$ for every interpretation $\mathcal{I}$. Due to Proposition 49, one of the following two cases holds: $R(a,b) \models_{\mathcal{T}} L_1$ or $R(a,b) \models_{\mathcal{T}} L_2$. Consider the first case (the second one is symmetric). Combining $R(a,b) \models_{\mathcal{T}} L_1$ and $\alpha \models L_1 \to \neg L_2$ we obtain that $\alpha \models R(a,b) \to \neg L_2$. Thus, α is of the form $\exists R \sqsubseteq \neg B$ or $\exists R^- \sqsubseteq \neg B$ for some basic concept B. Either case contradicts the fact that $(\mathcal{T} \cup \mathcal{A})$ is a *DL-Lite*$^{pr}_{\mathcal{R}}$ KB. Similarly, the case when $\mathcal{A} \models_{\mathcal{T}} \exists R(a)$ can be proved. □

Proof of Proposition 15.
$\Leftarrow$. Assume that $\mathcal{I}_1 \cup \mathcal{I}_2$ does not model $\mathcal{T}$. Then, there is an assertion $\alpha \in cl(\mathcal{T})$ such that $\mathcal{I}_1 \cup \mathcal{I}_2$ does not model α. If α is a PI, then there is a ground atom g_1 in $\mathcal{I}_1 \cup \mathcal{I}_2$ satisfying the property: for every ground atom g_2 such that $g_1 \to g_2$ is an instantiation of the first-order translation of α, it holds that $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$. The belonging $g_1 \in \mathcal{I}_1 \cup \mathcal{I}_2$ implies that one of the two cases holds: *(i)* $g_1 \in \mathcal{I}_1$ or *(ii)* $g_1 \in \mathcal{I}_2$. From Case (i) together with $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$ we conclude that $\mathcal{I}_1 \not\models \alpha$, and from Case (ii) together with $g_2 \notin \mathcal{I}_1 \cup \mathcal{I}_2$ we conclude that $\mathcal{I}_2 \not\models \alpha$. Either case contradicts the fact that $\mathcal{I}_1$ and $\mathcal{I}_2$ are models of $\mathcal{T}$. If α is a NI, then, due to Lemma 12 (more precisely, its straightforward extended to the case when $\mathcal{A}$ a possibly infinite set of atoms) of [8], there are two atoms f_1 and f_2 in $\mathcal{I}_1 \cup \mathcal{I}_2$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. Since $\mathcal{I}_1 \models \alpha$ and $\mathcal{I}_2 \models \alpha$, neither $\{f_1, f_2\} \subseteq \mathcal{I}_1$ nor $\{f_1, f_2\} \subseteq \mathcal{I}_2$ holds. Thus, either of the two cases holds: $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$ or $f_2 \in \mathcal{I}_1$ and $f_1 \in \mathcal{I}_2$. Either case contradicts the assumption of the if direction.

$\Rightarrow$. Assume that there are $f_1 \in \mathcal{I}_1$ and $f_2 \in \mathcal{I}_2$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$. That is, there are two ground atoms g_1 and g_2 and an NI α in $\mathcal{T}$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \{g_1, g_2\}$ and $g_1 \not\to g_2$ is an instantiation of the first-order translation of α. Moreover, due to Proposition 49, we can assume that

$$f_1 \models_{\mathcal{T}} g_1 \quad \text{and} \quad f_2 \models_{\mathcal{T}} g_2. \tag{B.2}$$

Since each of g_is is either unary or binary ground atoms and due to the semantics of a *DL-Lite*$_{\mathcal{R}}$ inclusion assertion, the following holds:

(i) both g_i share at least one constant, – let $\mathcal{S}$ be the set of these constants (clearly $\mathcal{S}$ has either one or two elements) – and due to Equation B.2 $\mathcal{S}$ is also shared by f_is and occurs in atoms of $\mathcal{I}_i$s; and

(ii) each g_i may have at most one constant that does not occur in the other atom g_j where $i \neq j$ – let $\mathcal{S}_i$ be the set of these constants for the corresponding g_i (clearly each $\mathcal{S}_i$ is either an empty set or a singleton).

Since $f_i \in \mathcal{I}_i$, $f_i \models_{\mathcal{T}} g_i$, and $\mathcal{I}_i$ is a model of $\mathcal{T}$, there is a renaming σ_i of constants of S_is that maps $\mathcal{S}_i$ to $adom(\mathcal{I}_i)$ and is the identity mapping on $adom(\mathcal{I}_i)$ such that $\sigma(g_i) \in \mathcal{I}_i$. Thus, $\{\sigma_1(g_1), \sigma_2(g_2)\} \subseteq \mathcal{I}_1 \cup \mathcal{I}_2$. On the other hand, $\sigma_1(g_1) \not\to \sigma_2(g_2)$ is an instantiation of the first-order interpretation of α, thus $\mathcal{I}_1 \cup \mathcal{I}_2$ does not satisfy α, which contradicts the assumption of the only-if direction. □

Proof of Proposition 16.
Assume that there is a general MA g such that $\mathcal{I} \setminus root_{\mathcal{T}}(g) \not\models \mathcal{T}$. Then, there is an assertion $\alpha \in cl(\mathcal{T})$ s.t.

$$\mathcal{I} \setminus root_{\mathcal{T}}(g) \not\models \alpha. \tag{B.3}$$

Assume that α is an NI. Clearly, if a set of atoms satisfies a negative inclusion assertion, then any subset of this set of atoms does so. This implies that, since $\mathcal{I} \models \alpha$ and $\mathcal{I} \setminus root_{\mathcal{T}}(g) \subseteq \mathcal{I}$, $\mathcal{I} \setminus root_{\mathcal{T}}(g) \models \alpha$, which contradicts the assumption in Equation B.3.

Assume that g is a positive MA and α is a PI. Then, Equation B.3 implies that there is a ground atom f_1 in $\mathcal{I} \setminus root_{\mathcal{T}}(g)$ satisfying the property: for every ground atom f_2 such that $f_1 \to f_2$ is an instantiation of the first-order translation of α, $f_2 \notin \mathcal{I} \setminus root_{\mathcal{T}}(g)$. Observe that $f_1 \in \mathcal{I}$ and $\mathcal{I} \models \alpha$, thus at least one such f_2, say

$\hat{f}_2$, is in $\mathcal{I}$. Since $\hat{f}_2 \notin \mathcal{I} \setminus root_{\mathcal{T}}(g)$, we have that $\hat{f}_2 \in root_{\mathcal{T}}(g)$. Therefore, by the definition of $root_{\mathcal{T}}(g)$, $f_1 \in root_{\mathcal{T}}(g)$ and $f_1 \notin \mathcal{I} \setminus root_{\mathcal{T}}(g)$, which contradicts the fact that $f_1 \in \mathcal{I} \setminus root_{\mathcal{T}}(g)$ and concludes the proof.

Assume that g is a negative MA and α is a PI. Let α be $B \sqsubseteq B'$. By exactly the same reason as the case of positive g, there are the atoms f_1 and $\hat{f}_2$ such that $f_1 \to \hat{f}_2$ instantiate $B \sqsubseteq B'$. Since $\hat{f}_2 \in root_{\mathcal{T}}(g)$, there is an NI of the form $B' \sqsubseteq \neg B''$ such that $\mathcal{T} \models B' \sqsubseteq \neg B''$ and $\hat{f}_2 \to g$ is its instantiation. From $\mathcal{T} \models B' \sqsubseteq \neg B''$ and $\mathcal{T} \models B \sqsubseteq B'$ we conclude that $\mathcal{T} \models B \sqsubseteq \neg B''$ and therefore $f_1 \in root_{\mathcal{T}}(g)$ which contradicts the fact that $f_1 \in \mathcal{I} \setminus root_{\mathcal{T}}(g)$ and concludes the proof. The case when α is $R \sqsubseteq R'$ is analogous. □

Proof of Proposition 17.
Let $\mathcal{K} = (\mathcal{T} \cup \mathcal{A})$ and $\mathcal{J}_0$ a finite model of $(\mathcal{T} \cup \mathcal{A})$. Due to finite model property of *DL-Lite*$_{\mathcal{R}}$, such $\mathcal{J}_0$ exists. Consider the following interpretation:

$$\mathcal{J} = \mathcal{J}_0 \cup (\mathcal{I} \setminus \mathcal{I}^{\mathcal{N}} \setminus \mathcal{I}^{\mathcal{J}}).$$

We now show that $\mathcal{J} \models \mathcal{T} \cup \mathcal{N}$. First, observe that $\mathcal{J} \models \mathcal{N}$ due to $\mathcal{J}_0 \models \mathcal{N}$ and the fact that $\mathcal{N}$ consists only of positive assertions of the form $R(a, b)$ and $A(c)$. We show that $\mathcal{J} \models \mathcal{T}$ by first proving that $\mathcal{J}_0 \models \mathcal{T}$ and $\mathcal{I} \setminus \mathcal{I}^{\mathcal{N}} \setminus \mathcal{I}^{\mathcal{J}} \models \mathcal{T}$, and then applying Proposition 15. $\mathcal{J}_0 \models \mathcal{T}$ obviously holds since $\mathcal{J}_0 \models \mathcal{T} \cup \mathcal{N}$. Let us denote $\mathcal{I}' = \mathcal{I} \setminus \mathcal{I}^{\mathcal{N}} \setminus \mathcal{I}^{\mathcal{J}}$. Since $\mathcal{I}$ satisfies all NIs of $cl(\mathcal{T})$, so does every subset of $\mathcal{I}$, including $\mathcal{I}'$. It remains to show that $\mathcal{I}'$ satisfies all PIs of $cl(\mathcal{T})$. Assume this is not the case. Thus, there is a PI $\alpha \in cl(\mathcal{T})$ and $g \in \mathcal{I}'$ such that g can instantiate the left-hand side of α and there is no atom $f \in \mathcal{I}'$ such that $g \to f$ is an instantiation of the first-order interpretation of α. Recall that $\mathcal{I} \models \alpha$, therefore there is $f' \in \mathcal{I}$ such that $g \to f'$ is an instantiation of the first-order interpretation of α. In particular, it means that $g \models_{\{\alpha\}} f'$. Since $f' \notin \mathcal{I}'$, it holds that $f' \in \mathcal{I}^{\mathcal{N}}$ or $f' \in \mathcal{I}^{\mathcal{J}}$. In the former case, $f' \in \mathcal{I}^{\mathcal{N}}$ implies that $\{f'\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. Combining $g \models_{\alpha} f'$ and $\{f'\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$, we obtain $\{g\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. Thus, $g \in \mathcal{I}^{\mathcal{N}}$ and $g \notin \mathcal{I}'$, which contradicts the assumption that $g \in \mathcal{I}'$. The case $f' \notin \mathcal{I}^{\mathcal{J}}$ is analogous. Thus, $\mathcal{I}' \models \alpha$ for every PI $\alpha \in cl(\mathcal{T})$, and therefore $\mathcal{I}' \models \mathcal{T} \cup \mathcal{N}$.

Observe that $\mathcal{I} \ominus \mathcal{J} = \mathcal{I}^{\mathcal{N}} \cup \mathcal{I}^{\mathcal{J}}$ which is a finite set. Therefore, for every $\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}$ it holds: $\mathcal{I} \ominus \mathcal{J}' \subseteq \mathcal{I} \ominus \mathcal{J}$ and therefore $\mathcal{I} \ominus \mathcal{J}'$ is finite. □

Proof of Proposition 19.
Let $g = R(a, b)$, the case when $g = \exists R(a)$ is analogous. Assume $\mathcal{A} \models_{\mathcal{T}} R(a, b)$, while there is $\mathcal{J}_0 \in \mathcal{S}$ such that $\mathcal{J}_0 \not\models R(a, b)$. Let $\mathcal{I}_0$ be a model of $\mathcal{T} \cup \mathcal{A}$ such that $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$ under $\mathcal{L}^a_{\#}$. We now exhibit $\mathcal{J}_0' \models \mathcal{T} \cup \mathcal{N}$ such that $|\mathcal{I}_0 \ominus \mathcal{J}_0'| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. Consider $\mathcal{J}_0' = \mathcal{J}_0 \cup \mathcal{I}_0^{R(a,b)}$. Note that $\mathcal{A} \models_{\mathcal{T}} R(a, b)$ and therefore the set $\mathcal{I}_0^{R(a,b)}$ is not empty.

Observe that $\mathcal{J}_0' \models \mathcal{T} \cup \mathcal{N}$. Indeed, $\mathcal{J}_0' \models \mathcal{N}$ since $\mathcal{N}$ contains positive MAs only and $\mathcal{J}_0 \models \mathcal{N}$. $\mathcal{J}_0'$ models all PIs from $\mathcal{T}$ since both $\mathcal{J}_0$ and $\mathcal{I}_0^{R(a,b)}$ does so. Assume there is an NI $\alpha \in cl(\mathcal{T})$ of the form $A_1 \sqsubseteq \neg A_2$, where A_1 and A_2 are atomic, such that $\mathcal{J}_0' \not\models \alpha$.[4] Thus, there is a pair of atoms $\{A_1(c), A_2(c)\} \subseteq \mathcal{J}_0'$. Observe that $\{A_1(c), A_2(c)\} \not\subseteq \mathcal{I}_0^{R(a,b)}$ and $\{A_1(c), A_2(c)\} \not\subseteq \mathcal{J}_0$. Indeed, the ABox $\{R(a, b)\}$ obviously satisfies α and due to Lemma 12 of [5] so does the model $\mathcal{I}_0^{R(a,b)}$, and therefore $\{A_1(c), A_2(c)\} \not\subseteq \mathcal{I}_0^{R(a,b)}$. Since $\mathcal{J}_0 \in \mathcal{S}$, it clearly holds that $\{A_1(c), A_2(c)\} \not\subseteq \mathcal{J}_0$. Therefore, one of the two cases holds: $A_1(c) \in \mathcal{J}_0$ and $A_2(c) \in \mathcal{I}_0^{R(a,b)}$, or $A_2(c) \in \mathcal{J}_0$ and $A_1(c) \in \mathcal{I}_0^{R(a,b)}$. Either case

[4] Note that $\mathcal{T} \cup \mathcal{A}$ is a *DL-Lite*$^{pr}_{\mathcal{R}}$ KB and therefore all NIs in $cl(\mathcal{T})$ has only atomics concepts on the left and the right of $\sqsubseteq$.

is possible since neither $\mathcal{J}_0$ nor $\mathcal{I}_0^{R(a,b)}$ is empty. Consider the first case, the second case is symmetric. The membership $A_2(c) \in \mathcal{I}_0^{R(a,b)}$ implies an existence of a sequence of atoms $f_1, \ldots, f_n$ in $\mathcal{I}_0^{R(a,b)}$ such that $n \geq 2$, $f_1 = R(a,b)$, $f_n = A_2(c)$, and for each $1 \leq i \leq (n-1)$ there is a PI $\alpha_1 \in \mathit{cl}(\mathcal{T})$ such that $f_i \to f_{i+1}$ is an instantiation of α_i. We now show by induction on n that $\mathit{cl}(\mathcal{T})$ contains an NI of the form $\exists R' \sqsubseteq \neg A'$ for some role R' and atomic concept A', which will give a contradiction with the fact that $\mathcal{T} \cup \mathcal{A}$ is a *DL-Lite*$_{\mathcal{R}}^{pr}$ KB. If $n = 2$ then $\alpha_1 = \exists R \sqsubseteq A_2$ or $\alpha_1 = \exists R^- \sqsubseteq A_2$. The former case combined with α gives that $\exists R \sqsubseteq \neg A_1 \in \mathit{cl}(\mathcal{T})$ and the latter one: $\exists R^- \sqsubseteq \neg A_1 \in \mathit{cl}(\mathcal{T})$. Thus, we obtain a contradiction. If $n > 2$, then consider α_{n-1}. The shape of α_{n-1} is either $A' \sqsubseteq A_2$ or $\exists R' \sqsubseteq A_2$. Combining the former case with α, we obtain that $A_1 \sqsubseteq \neg A'$ and we conclude the proof by the induction assumption. Combining the later case with α we obtain that $\exists R' \sqsubseteq \neg A_1$, which gives a contradiction. We conclude that $\mathcal{J}_0' \models \mathcal{T} \cup \mathcal{N}$.

It remains to show that $|\mathcal{I}_0 \ominus \mathcal{J}_0'| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. By contraction, $\mathcal{I}_0 \ominus \mathcal{J}_0' \subsetneq \mathcal{I}_0 \ominus \mathcal{J}_0$. One can show that $\mathcal{I}_0 \ominus \mathcal{J}_0'$ is finite using the same argument as in Proposition 17. Thus, $|\mathcal{I}_0 \ominus \mathcal{J}_0'| < |\mathcal{I}_0 \ominus \mathcal{J}_0|$. □

Appendix C. Proofs for Section 4.2

Proposition 50.

Proof of Proposition 21.
Due to Proposition 15, it suffices to show that for every $f_1 \in (\mathcal{I} \setminus \mathit{root}_{\mathcal{T}}(\neg A(c)))$ and $f_2 \in \mathcal{J}[A(c)]$ we have $\{f_1, f_2\} \not\models_{\mathcal{T}} \bot$. Assume this is not the case, that is, there is $f_1 \in (\mathcal{I} \setminus \mathit{root}_{\mathcal{T}}(\neg A(c)))$ and $f_2 \in \mathcal{J}[A(c)]$ such that $\{f_1, f_2\} \models_{\mathcal{T}} \bot$.

We now show that

$$\{f_1, A(c)\} \models_{\mathcal{T}} \bot. \tag{C.1}$$

From $\{f_1, f_2\} \models_{\mathcal{T}} \bot$ it clearly follows that there is an NI of the form $A_1 \sqsubseteq \neg A_2$ such that $\mathcal{T} \models A_1 \sqsubseteq \neg A_2$, and there are atoms $A_1(d) \in (\mathcal{I} \setminus \mathit{root}_{\mathcal{T}}(\neg A(c)))[f_1]$ and $A_2(d) \in \mathcal{J}[A(c)]$. We do not need to consider NIs that involve roles since we are in the case of *DL-Lite*$_{\mathcal{R}}^{pr}$. From $A_2(d) \in \mathcal{J}[A(c)]$ we conclude that $d = c$. Indeed, from $A_2(d) \in \mathcal{J}[A(c)]$ we conclude that there is a sequence of atoms $f_1, \ldots, f_n$, where *(i)* $f_1 = A(c)$; *(ii)* $f_n = A_2(d)$, *(iii)* each f_{i+1} is derivable from f_i by triggering a positive inclusion assertion α_i of $\mathit{cl}(\mathcal{T})$. If $c \neq d$, then there is a role symbol occurring in at least one α_i. Indeed, if each α_i has no role symbol, then due to transitivity of $\models_{\mathcal{T}}$, we have $f_1 \models_{\mathcal{T}} f_n$ and therefore $c = d$. Let R be a role symbol occurring in α_j with the highest index, that is, $j = n$, or $j < n$ and for each α_i where $j < i < n$ there is no role symbol occurring in α_i. Then, α_j is of the form $\exists R \sqsubseteq A'$. Thus, $\mathcal{T} \models_{\mathcal{T}} \exists R \sqsubseteq A_2$. Combining this with $\mathcal{T} \models A_1 \sqsubseteq \neg A_2$, we obtain that $\mathcal{T} \models_{\mathcal{T}} \exists R \sqsubseteq \neg A_1$, which contradicts the fact that $\mathcal{T}$ is in *DL-Lite*$_{\mathcal{R}}^{pr}$. Thus, there are no role symbols in each α_t, where $1 \leq t \leq n$. Therefore, $c = d$ and $A(c) \models_{\mathcal{T}} A_2(c)$.

Analogously, one can show that $A_1(c) \in (\mathcal{I} \setminus \mathit{root}_{\mathcal{T}}(\neg A(c)))[f_1]$ implies that $f_1 \models_{\mathcal{T}} A_1(c)$. To sum up, we proved that

$$f_1 \models_{\mathcal{T}} A_1(c), \quad A(c) \models_{\mathcal{T}} A_2(c), \quad A_1(c) \models_{\{A_1 \sqsubseteq \neg A_2\}} \neg A_2(c),$$

thus Equation C.1 holds.

Since for every $f_2 \in \mathcal{J}[A(c)]$ it holds that $A(c) \models_{\mathcal{T}} f_2$, we have $\{f_1, A(c)\} \models_{\mathcal{T}} \bot$. Thus, $\{f_1, A(c)\} \models_{\mathcal{T}} \neg A(c)$. There are only two literals in $\{f_1, A(c)\}$ and $A(c) \not\models_{\mathcal{T}} \neg A(c)$. Thus, due to Proposition 49, we conclude that $f_1 \models_{\mathcal{T}} \neg A(c)$. This contradicts the assumption that $f_1 \in (\mathcal{I} \setminus \mathit{root}_{\mathcal{T}}(\neg A(c)))$ and concludes the proof. □

Appendix D. Proofs for Section 5.1

Proof of Proposition 29.
Recall that

$$\mathsf{func}(R, c) \doteq \forall x \forall y. (R(x, c) \wedge R(x, y) \rightarrow y = c) .$$

Assume there is a *DL-Lite*$_{\mathcal{R}}$ KB $\mathcal{K} = (\mathcal{T} \cup \mathcal{A})$ such that $\mathcal{K} \not\models \neg\exists R^-(c)$ and $\mathcal{K} \models \mathsf{func}(R, c)$. Let a and b be constants *not* occurring in $\mathcal{K}$. Consider $\mathcal{A}' = \mathcal{A} \cup \{R(a, c), R(a, b)\}$. We now show that $\mathcal{A}'$ satisfies all the NIs of *cl*($\mathcal{T}$). If this is the case, then due to Lemma 12 of [5], this observation gives that $\mathcal{T} \cup \mathcal{A}'$ is satisfiable. Let $\mathcal{I}$ be a model of $\mathcal{T} \cup \mathcal{A}'$. Since $\mathcal{I} \models \mathcal{A}'$, it holds that $\mathcal{I} \models \mathcal{A}$ and therefore $\mathcal{I} \models \{R(a, c), R(a, b)\}$, that is, $\mathcal{I}$ does not satisfy $\mathsf{func}(R, c)$. On the other hand, $\mathcal{I} \models \mathcal{K}$. This contradicts the fact that $\mathcal{K} \models \mathsf{func}(R, c)$.

So, it remains to show that $\mathcal{A}'$ satisfies all the NIs of *cl*($\mathcal{T}$). Assume there is an NI $\alpha \in$ *cl*($\mathcal{T}$) such that $\mathcal{A}'$ does not satisfy α. Then, there are two MAs f and g in $\mathcal{A}'$ such that $f \rightarrow \neg g$ is an instantiation of the first-order interpretation of α. Four cases are possible: *(i)* $\{f, g\} \subseteq \{R(a, c), R(a, b)\}$, *(ii)* $f \in \mathcal{A}$ and $g \in \{R(a, c), R(a, b)\}$, *(iii)* $g \in \mathcal{A}$ and $f \in \{R(a, c), R(a, b)\}$, and *(iv)* $\{f, g\} \in \mathcal{A}$. Assume that Case (i) holds. One possibility of $\{f, g\} \in \{R(a, c), R(a, b)\}$ is when $f = R(a, c)$ and $g = R(a, b)$. Then, $\alpha = \exists R \sqsubseteq \neg\exists R$, which contradicts the coherency of $\mathcal{K}$. Other possibilities of $\{f, g\} \in \{R(a, c), R(a, b)\}$ are analogous. Assume that Case (ii) holds. Let $g = R(a, b)$. Since $f \rightarrow \neg g$ instantiates α, f and g should share at least one constant. Since $f \in \mathcal{A}$ and neither a nor b occurs in $\mathcal{A}$, this constant is c. Hence, $g = R(a, c)$. Therefore, α is of the form $B \sqsubseteq \neg R^-$, and $B(c) \in \mathcal{A}$. Thus, $\mathcal{K} \models \neg\exists R^-(c)$, which contradicts the assumptions of the proposition on $\mathcal{K}$. Case (iii) is analogous to Case (ii). Assume that Case (iv) holds. Thus, $\mathcal{A} \not\models \alpha$ and due to Lemma 12 of [5] $\mathcal{T} \cup \mathcal{A}$ is unsatisfiable, which contradicts the coherence of $\mathcal{K}$. □

Appendix E. Proofs for Section 5.2

Proof of Proposition 36.
Follow from the correctness of `BP`, see Theorem 39. □

Proof of Proposition 40.
To see that $\mathcal{I}_0 \models \mathcal{K}$ observe that, by the definition of $\mathcal{A}_1$, the set $\mathcal{A} \cup \{R_a(a, b_a)\}$ satisfies all the NIs in *cl*($\mathcal{T}$). Moreover, $\{R_a(a, b_a), R_{a'}(a', b_{a'})\}$, where a and a' are such that $A(a)$ and $A'(a')$ are in $\mathcal{A}_1$ for some concepts A and A', satisfies all the NIs in *cl*($\mathcal{T}$). Thus, the set of MAs that is chased in Equation (15) satisfies all the NIs in *cl*($\mathcal{T}$). Hence, due to Lemma 12 of [5], $\mathcal{I}_0 \models \mathcal{T}$. The fact that $\mathcal{I}_0 \models \mathcal{A}$ is trivial.

To see that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{K}$ observe that for each i the set $\{R_i(x_i, d), A_i(x_i)\}$ satisfies all the NIs in *cl*($\mathcal{T}$), so does its chase with $\mathcal{T}$. Since x_is are fresh and $\mathcal{K}, \mathcal{N}$ is a simple evolution setting, i.e., no disjointness between roles is allowed, for any $g_1 \in \mathit{chase}_{\mathcal{T}}(\{R_i(x_i, d), A_i(x_i)\})$ and $g_2 \in \mathit{chase}_{\mathcal{T}}(\{R_j(x_j, d), A_j(x_j)\})$, it holds that $\{g_1, g_2\} \not\models_{\mathcal{T}} \bot$. Therefore, we can apply Proposition 15 to the union of chases over $1 \leq i \leq |\mathcal{D}|$ and conclude that it satisfies $\mathcal{T}$. Clearly, for each $g_1 \in \mathcal{I}^{can}$ and g_2 in the union of chases, $\{g_1, g_2\} \not\models_{\mathcal{T}} \bot$ and again by applying Proposition 15 we conclude that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{T}$. The fact that $\mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \models \mathcal{A}$ trivially follows from the fact that $\mathcal{I}^{can} \models \mathcal{A}$.

The proof of $\mathcal{J}_0 \in \mathcal{I}_0 \diamond \mathcal{N}$ and $\mathcal{J}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \in \mathcal{I}[\mathcal{D}, \mathcal{R}, \mathcal{B}] \diamond \mathcal{N}$, is straightforward by the definition of `BZP` and `BP` procedures. □

Proposition 51. *Let $n \geq 2$ be a natural number, $\alpha_1 = B_1 \sqsubseteq B'_1, \ldots, \alpha_n = B_n \sqsubseteq B'_n$ DL-Lite$_{\mathcal{R}}$ PIs, $\beta = B \sqsubseteq \neg B'$ a DL-Lite$_{\mathcal{R}}$ NI, and $g_0, \ldots, g_n, f$ a sequence of ground atoms such that $g_{i-1} \rightarrow g_i$ for $1 \leq i \leq n$ instantiates α_i, and $g_n \rightarrow \neg f$ instantiates β. If $B_1 \sqsubseteq \neg B' \notin cl(\{\alpha_1, \cdots, \alpha_n, \beta\})$, then at least one of the following conditions holds*

(i) there is $1 \le i \le n-1$ *such that* α_i *is* $B_i \sqsubseteq \exists R$ *and* α_{i+1} *is* $\exists R^- \sqsubseteq B'_{i+1}$*, where* $R \in \Sigma(\mathcal{T})$*.*
(ii) $B'_n = \exists R$ *and* $B = \exists R^-$*, where* $R \in \Sigma(\mathcal{T})$*.*

Proof. We prove it by induction on n. Assume $n = 1$, thus, g_1 instantiates B'_1 and B, and therefore B'_1 and B share the predicate symbol (concept or role name). Case (i) is not applicable, since there is only one α_i. Assume that Case (ii) does not hold. Then, one of the following options holds: *(1)* either B'_1 is (of the form) $\exists R$ and B is $\exists R$ or *(2)* B'_1 is A and B is A. From either case we conclude that $B_1 \sqsubseteq \neg B' \in cl(\{\alpha_1, \beta\})$, which contradicts the assumption of the proposition.

Assume $n > 1$. If Case (i) does not hold, then consider α_i and α_{i+1} for some $i < n$. Since g_i instantiates B'_i and B_{i+1}, they share the predicate symbol, thus one of the following options hold: *(1)* either B'_i is $\exists R$ and B_{i+1} is $\exists R$ or *(2)* B'_i is A and B_{i+1} is A. Thus, as in the case above, we conclude that $B_i \sqsubseteq B'_{i+1} \in cl(\{\alpha_i, \alpha_i + 1\})$. Applying this argument iteratively to pairs of α_i and α_{i+1} from $i = 1, \ldots, n-1$, we obtain that $B_1 \sqsubseteq B'_n \in \{\alpha_1, \ldots, \alpha_n\}$. If Case (ii) does not hold, then, using the same argument as in the case of $n = 1$ to $B_1 \sqsubseteq B'_n$ and $B \sqsubseteq B'$, we obtain that $B_1 \sqsubseteq \neg B' \in \{\alpha_1, \ldots, \alpha_n, \beta\}$, which contradicts the assumption of the proposition. □

Proposition 52. *Let* $\mathcal{K} = (\mathcal{T} \cup \mathcal{A})$ *and* $\mathcal{N}$ *be a simple evolution setting,* $\mathcal{I} \models \mathcal{K}$ *and* $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ *under* $\mathcal{L}^a_{\subseteq}$*. If* $\mathcal{N} \models_{\mathcal{T}} \exists R(a)$*,* $\mathcal{N} \not\models_{\mathcal{T}} R(a,b)$*, and there is an NI* $\alpha \in cl(\mathcal{T})$ *such that* $\mathcal{I} \cup \{R(a,b)\} \not\models \alpha$*, then* $R(a,b) \notin \mathcal{J}$*.*

Proof. Assume $R(a,b) \in \mathcal{J}$, then consider $\mathcal{J}' = \mathcal{J} \setminus \{R(a,b)\}$. We now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, thus contradicting the fact that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Since $\mathcal{N} \not\models_{\mathcal{T}} R(a,b)$, it clearly holds that $\mathcal{J}' \models \mathcal{N}$. Since $\mathcal{N} \models_{\mathcal{T}} \exists R(a)$ and the evolution setting is simple, there is c such that $\mathcal{N} \models_{\mathcal{T}} R(a,c)$ Thus, $R(a,c) \in \mathcal{J}'$ and therefore $\mathcal{J}'$ satisfies PIs of $cl(\mathcal{T})$. Since $\mathcal{J}' \subseteq \mathcal{J}$ and $\mathcal{J}$ satisfies NIs of $cl(\mathcal{T})$, so does $\mathcal{J}'$. We conclude that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$.

Finally, observe that the fact that $\mathcal{I} \cup \{R(a,b)\} \not\models \alpha$ implies that $R(a,b) \notin \mathcal{I}$. Taking into account that $R(a,b) \notin \mathcal{J}'$, we conclude that $R(a,b) \notin \mathcal{I} \ominus \mathcal{J}'$. At the same time $R(a,b) \in \mathcal{J}$ and therefore $R(a,b) \in \mathcal{I} \ominus \mathcal{J}$. Thus, $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ holds and we conclude the proof. □

Proof of Proposition 41.
Let $\mathcal{K} = (\mathcal{T} \cup \mathcal{A})$ and $\mathcal{S} = \mathit{Align}_{\mathcal{T}}((\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}), \mathcal{N})$. Assume there exists a model $\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$ such that $\mathcal{S} \not\subseteq \mathcal{J}'$. Consider $\mathcal{J}'' = \mathcal{J}' \cup \mathcal{S}$. We will show that $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}'' \subsetneq \mathcal{I} \ominus \mathcal{J}'$ which yields a contradiction with $\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$.

To see that $\mathcal{I} \ominus \mathcal{J}'' \subsetneq \mathcal{I} \ominus \mathcal{J}'$ observe the following:

$$\begin{aligned}
\mathcal{I} \ominus \mathcal{J}'' &= (\mathcal{I} \setminus \mathcal{J}'') \cup (\mathcal{J}'' \setminus \mathcal{I}) \\
&= ((\mathcal{I} \setminus \mathcal{J}') \setminus \mathcal{S}) \cup \big((\mathcal{J}' \setminus \mathcal{I}) \cup (\mathcal{S} \setminus \mathcal{I})\big) \\
&\subsetneq_{\text{due to } \mathcal{S} \subseteq \mathcal{I} \text{ and } \mathcal{S} \not\subseteq \mathcal{J}'} (\mathcal{I} \setminus \mathcal{J}') \cup (\mathcal{J}' \setminus \mathcal{I}) \cup (\mathcal{S} \setminus \mathcal{I}) \\
&=_{\text{due to } \mathcal{S} \subseteq \mathcal{I}} (\mathcal{I} \setminus \mathcal{J}') \cup (\mathcal{J}' \setminus \mathcal{I}) = \mathcal{I} \ominus \mathcal{J}'.
\end{aligned}$$

It remains to show that $\mathcal{J}'' \models \mathcal{T} \cup \mathcal{N}$. The fact that $\mathcal{J}'' \models \mathcal{N}$ follows trivially from the fact that $\mathcal{J}' \in \mathit{Mod}(\mathcal{N})$, $\mathcal{J}' \subseteq \mathcal{J}''$ and that each assertion of $\mathcal{N}$ is of the form $R(a,b)$ or $A(c)$. We now prove that $\mathcal{J}'' \models \mathcal{T}$ by showing that both $\mathcal{S}$ and $\mathcal{J}'$ are models of $\mathcal{T}$ and then by applying Proposition 15. Obviously, $\mathcal{J}' \models \mathcal{T}$ holds by the definition of $\mathcal{J}'$. Observe that by the definition of alignment:

$$\mathcal{S} = \big(\mathcal{I} \setminus \mathcal{B}_{\mathcal{I}}\big) \setminus \bigcup_{g \in \mathcal{I} \setminus \mathcal{B}_{\mathcal{I}} \text{ s.t. } \{g\} \cup \mathcal{N} \models_{\mathcal{T}} \bot} \mathit{root}_{\mathcal{T}}(g).$$

Since $\mathcal{I} \models \mathcal{T}$, one can show that $\mathcal{S} \models \mathcal{T}$ by applying Proposition 16 a necessary (probably infinite) number of times: first to $\mathcal{B}_\mathcal{I}$ and then to each $g \in \mathcal{I} \setminus \mathcal{B}_\mathcal{I}$ s.t. $\{g\} \cup \mathcal{N} \models_\mathcal{T} \bot$.

Since we proved that $\mathcal{S} \models \mathcal{T}$ and $\mathcal{J}' \models \mathcal{T}$ we can apply Proposition 15, that is, $\mathcal{S} \cup \mathcal{J}' \models \mathcal{T}$ if for every $f \in \mathcal{S}$ and $g \in \mathcal{J}'$ it holds: $\{f, g\} \not\models_\mathcal{T} \bot$. Assume this is not the case, and there are $f \in \mathcal{S}$ and $g \in \mathcal{J}'$ such that $\{f, g\} \models_\mathcal{T} \bot$. Let $\mathcal{G}$ (resp. $\mathcal{H}$) be the set of atoms g of $\mathcal{J}'$ such that $\{f, g\} \models_\mathcal{T} \bot$ for some $f \in \mathcal{S}$ and $root_\mathcal{T}^{\mathcal{J}'}(g) \cap \mathcal{N} = \emptyset$ (resp. $root_\mathcal{T}^{\mathcal{J}'}(g) \cap \mathcal{N} \neq \emptyset$). By our assumption, $\mathcal{H} \cup \mathcal{G} \neq \emptyset$. Note that it is enough to consider only the case when f is a unary atom. Indeed, if f is binary, i.e. if $f = R(a, b)$, then, due to the fact that in *DL-Lite*$_\mathcal{R}$ disjointness is allowed between basic concepts only, $\{R(a, b), g\} \models_\mathcal{T} \bot$ holds if and only if either $\{\exists R(a), g\} \models_\mathcal{T} \bot$ or $\{\exists R^-(b), g\} \models_\mathcal{T} \bot$. If the first case holds, then we can introduce a fresh concept name $A_{\exists R}$ and extend $\mathcal{I}$ by assigning $A_{\exists R}^\mathcal{I} = (\exists R)^\mathcal{I}$ to be the interpretation of $R^\mathcal{I}$ projected on the fist coordinate. Then, both the original $\mathcal{I}$ and the extended one will behave equivalently wrt to the proposition.

We first show that $\mathcal{H} = \emptyset$. Assume this is not the case and there is $g \in \mathcal{H}$. Let $g' \in root_\mathcal{T}^{\mathcal{J}'}(g) \cap \mathcal{N}$. By the definition of $root_\mathcal{T}$ for models, there is a sequence of PIs $\alpha_1 = B_1 \sqsubseteq B_1', \ldots, \alpha_n = B_n \sqsubseteq B_n'$ in $cl(\mathcal{T})$ and of atoms $g_0, \ldots, g_n$ in $root_\mathcal{T}^{\mathcal{J}'}(g)$ such that $g_0 = g'$, $g_n = g$ and $g_{i-1} \to g_i$ for $1 \leq i \leq n$ instantiates α_i. From $\{f, g\} \models_\mathcal{T} \bot$ and Lemma 12 of [5] it follows that there is an NI $\beta = B \sqsubseteq \neg B'$ in $cl(\mathcal{T})$ such that $g_n \to \neg f$ instantiates β. We now show that $B_1 \sqsubseteq \neg B' \notin cl(\{\alpha_1, \ldots, \alpha_n\})$ holds and then apply Proposition 51. Assume $B_1 \sqsubseteq \neg B' \in cl(\{\alpha_1, \ldots, \alpha_n\})$, then it holds that $\mathcal{T} \models B_1 \sqsubseteq \neg B'$. Moreover, $g_0 \to \neg f$ instantiates $B_1 \sqsubseteq \neg B'$ and therefore, $\{f, g_0\} \models_{\{B_1 \sqsubseteq \neg B'\}} \bot$. Combining $\{f, g_0\} \models_{\{B_1 \sqsubseteq \neg B'\}} \bot$ and $\mathcal{T} \models B_1 \sqsubseteq \neg B'$, we conclude that $\{f, g_0\} \models_\mathcal{T} \bot$. Taking into account that $g_0 \in \mathcal{N}$, we finally conclude that $\{f\} \cup \mathcal{N} \models_\mathcal{T} \bot$. Therefore, $f \notin \mathcal{S}$ which gives a contradiction with $f \in \mathcal{S}$. Thus, we can apply Proposition 51.

Assume that Case (ii) of Proposition 51 holds, that is $\alpha_n = B_n \sqsubseteq \exists R$ and $\beta = \exists R^- \sqsubseteq \neg B'$. In particular, this means that g is of the form $R(x, y)$. Since the evolution setting is simple, $\mathcal{T} \not\models \exists R_1 \sqsubseteq \exists R$ for any role R_1. Combining this with $\alpha_n = B_n \sqsubseteq \exists R$, we obtain that B_n and all B_i and B_i' occurring in α_i for $1 \leq i \leq n-1$ are atomic concepts, say A_i and A_i', respectively. Indeed, let B_k' be of the form $\exists R_1^-$ and has the highest index among B_is with this property. Then, $\mathcal{T} \models \exists R_1^- \sqsubseteq \exists R$, which contradiction the fact that $\mathcal{K}, \mathcal{N}$ is a simple setting. This implies that $\{\alpha_1, \ldots, \alpha_n\} \models_\mathcal{T} A_1 \sqsubseteq \exists R$. Combining this with the our assumption that g' instantiates A_1 and $g = R(x, y)$, we obtain that $g' = A(x)$ and $A_1(x) \to R(x, y)$ instantiates $A_1 \sqsubseteq \exists R$. Since $g' \in \mathcal{N}$, we conclude that $\mathcal{N} \models_\mathcal{T} \exists R(x)$. Recall that $\{f, R(x, y)\} \models_\mathcal{T} \bot$ and $f \in \mathcal{I}$ (since $f \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{I}$), thus $\mathcal{I} \cup \{R(x, y)\}$ does not satisfy at least one NI of $cl(\mathcal{T})$. Now if $\mathcal{N} \not\models_\mathcal{T} R(x, y)$ holds, then we are in the conditions of Proposition 52 and can conclude that $R(x, y) \notin \mathcal{J}'$, which contradicts the fact that $R(x, y) = g \in \mathcal{J}'$. Therefore, $\mathcal{N} \models_\mathcal{T} R(x, y)$. Combining this with $\{f, R(x, y)\} \models_\mathcal{T} \bot$, we obtain that $\{f\} \cup \mathcal{N} \models_\mathcal{T} \bot$. Since f is unary we conclude that $f \in \mathcal{B}_\mathcal{I}$ which contradicts the fact that $f \in \mathcal{S}$.

Assume that Case (i) of Proposition 51 holds but Case (ii) does not. Then, let k be the maximal index satisfying that α_k and α_{k+1} are respectively of the form $B_k \sqsubseteq \exists R$ and $\exists R^- \sqsubseteq B_{k+1}'$. If $k = n-1$, then $\alpha_n = \exists R^- \sqsubseteq B_n'$. Moreover, since Case (ii) of Proposition 51 does *not* hold and in the evolution settings the entailment $\mathcal{T} \models \exists R^- \sqsubseteq \exists R'$ is not possible for any role R', we have that $B_i = A_i$ and $B_i' = A_i'$ (in fact, it even holds that $B_i = A_i$ and $B_i' = A_{i+1}$) for $1 \leq i \leq n-2$, $B_{n-1} = A_{n-1}$, $B_n' = A$, $B = A$ and $B' = A'$, where all A_j, A_j' and A, A' are from $\Sigma(\mathcal{T} \cup \mathcal{N})$. Thus,

$$\alpha_{n-1} = A_{n-1} \sqsubseteq \exists R, \;\; \alpha_n = \exists R^- \sqsubseteq A, \;\; g_0 = A_1(x), \;\; g_{n-1} = R(x, y), \;\; g_n = A(y) \text{ and } f = A'(y)$$

If $\mathcal{N} \models_\mathcal{T} R(x, y)$, then $\mathcal{N} \models_\mathcal{T} A'(y)$, and we obtain that $\{f\} \cup \mathcal{N} \models_\mathcal{T} \bot$, which contradicts $f \in \mathcal{S}$. If $\mathcal{N} \not\models_\mathcal{T} R(x, y)$, then due to $R(x, y) \models_{\{\alpha_n\}} A(y)$ and the fact that $\{A(y), A'(y)\}$ violates β, we conclude that $\{R(x, y), A'(y)\}$ violates $\exists R^- \sqsubseteq \neg A' \in cl(\mathcal{T})$. Thus, $\mathcal{I} \cup \{R(x, y)\}$ violates $\exists R^- \sqsubseteq \neg A'$ and we can apply Proposition 52 to conclude that that $R(x, y) \notin \mathcal{J}'$, which contradicts the fact that $R(x, y) \in root_{\mathcal{T}}^{\mathcal{J}'}(g)$. If $1 \leq k < n-1$, then analogously to the previous case we can show that $B'_k = \exists R\ B_{k+1} = \exists R^-$, for each $1 \leq i \leq k-2$ and $k+1 \leq i \leq n$ it holds that $B_i = A_i$ and $B'_i = A'_i$ (in fact, it even holds that $B'_i = A_{i+1}$) and also $B = A'_n$ and $B' = A'$, where all A_j, A'_j, A' and R are from $\Sigma(\mathcal{T} \cup \mathcal{N})$. Moreover,

$$\alpha_k = A_k \sqsubseteq \exists R,\ \ \alpha_{k+1} = \exists R^- \sqsubseteq A_{k+2},\ \ g_0 = A_1(x),\ \ g_k = R(x, y),\ \ g_n = A_n(y) \text{ and } f = A'(y).$$

Thus, applying the same reasoning as above we obtain a contradiction either with $f \in \mathcal{S}$ or $g \in \mathcal{J}'$. We conclude that $\mathcal{H} = \emptyset$ and $\mathcal{G} \neq \emptyset$.

Now consider

$$\hat{\mathcal{J}}' = \mathcal{J}' \setminus \bigcup_{g \in \mathcal{G}} root_{\mathcal{T}}(g) \cup \bigcup_{h \in root_{\mathcal{T}}^{\mathcal{J}'}(g) \text{ s.t. } g \in \mathcal{G}, h \in \mathcal{S}} \mathcal{S}[h].$$

We now show that $\hat{\mathcal{J}}' \ominus \mathcal{I} \subsetneq \mathcal{J}' \ominus \mathcal{I}$ and $\hat{\mathcal{J}}' \models \mathcal{T} \cup \mathcal{N}$, which contradicts the fact that $\mathcal{J}' \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. The inclusion $\hat{\mathcal{J}}' \ominus \mathcal{I} \subseteq \mathcal{J}' \ominus \mathcal{I}$ follows from the fact that each $\mathcal{S}[h] \subseteq \mathcal{I}$. The inclusion is strict since $\mathcal{G} \neq \emptyset$, $\mathcal{G} \subseteq \mathcal{J}'$, and $\mathcal{G} \cap \mathcal{I} = \emptyset$. Since for each $g \in \mathcal{G}$ it holds that $root_{\mathcal{T}}^{\mathcal{J}'}(g) \cap \mathcal{N} = \emptyset$, we have that $\hat{\mathcal{J}}' \models \mathcal{N}$. To see that $\hat{\mathcal{J}}' \models \mathcal{T}$, observe that $\mathcal{J}' \setminus \cup_{g \in \mathcal{G}} root_{\mathcal{T}}(g) \models \mathcal{T}$ due to Proposition 16, and clearly $\cup_{h \in root_{\mathcal{T}}^{\mathcal{J}'}(g) \text{ s.t. } h \in \mathcal{S}} \mathcal{S}[h] \models \mathcal{T}$. Therefore, we can apply Proposition 15: assume there is $g' \in \mathcal{J}' \setminus \cup_{g \in \mathcal{G}} root_{\mathcal{T}}(g)$ and $f' \in \mathcal{S}[h]$ for some h such that $\{g', f'\} \models_{\mathcal{T}} \bot$. Since $g' \in \mathcal{J}'$ and $f' \in \mathcal{S}$, one of the two options should hold: either $g' \in \mathcal{G}$ or $g' \in \mathcal{H}$. The first option is impossible since $g' \in \mathcal{J}' \setminus \cup_{g \in \mathcal{G}} root_{\mathcal{T}}(g)$, and therefore $g' \notin \mathcal{G}$. The second option is also impossible since $\mathcal{H} = \emptyset$. Thus, $\hat{\mathcal{J}}' \models \mathcal{T}$, we obtain a contradiction with $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$, hence $\mathcal{G} = \emptyset$ and we conclude the proof. □

Proof of Proposition 42.
Assume $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ and $\mathcal{J}' \in \mathcal{I}' \diamond \mathcal{N}$. Then, $\mathcal{I}$ is of the form Equations 14 or 15. Now one can conclude the proof by applying Proposition 41 to $\mathcal{I}$ and $\mathcal{I}'$. □

Proposition 53. *Let $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, $\mathcal{N}$ be a simple evolution setting, $\mathcal{I} \models \mathcal{K}$ and $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Then,*
(i) if $\mathcal{N} \not\models_{\mathcal{T}} A(a)$ and there is an NI $\alpha \in cl(\mathcal{T})$ such that $\mathcal{I} \cup \{A(a)\} \not\models \alpha$, then $A(a) \notin \mathcal{J}$.
(ii) if $\mathcal{N} \not\models_{\mathcal{T}} \exists R(a)$ and $\mathcal{N} \not\models_{\mathcal{T}} \exists R^-(b)$ and there is an NI $\alpha \in cl(\mathcal{T})$ such that $\mathcal{I} \cup \{R(a, b)\} \not\models \alpha$, then $R(a, b) \notin \mathcal{J}$.

Proof. Analogous to the proof of Proposition 52. □

Proof of Proposition 43.
Case (i): Let $\mathcal{I} \models \mathcal{K}$ be such that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Assume that $D(c) \notin \mathcal{J}$, while

$$\{R(x, c), A(x)\} \not\subseteq \mathcal{J}\ \forall x \in \Delta, \forall R \in \mathit{TR} \text{ s.t. } D(c) \in \texttt{DjnAts}[\mathcal{K}, \mathcal{N}](R) \text{ and } \forall A \in \texttt{ISubCon}[\mathcal{T}](\exists R). \tag{E.1}$$

Observe that the condition $\{R(x, c), A(x)\} \not\subseteq \mathcal{J}$ is satisfied when $R(x, c) \in \mathcal{J}$ and $A(x) \notin \mathcal{J}$. Let

$$\mathcal{D} = \{R(x, c) \mid x \in \Delta,\ R(x, c) \in \mathcal{J},\ D(c) \in \texttt{DjnAts}[\mathcal{K}, \mathcal{N}](R)\}.$$

Assume $\mathcal{D} \neq \emptyset$. Consider $root_{\mathcal{T}}^{\mathcal{J}}(R(x, c))$. Due to the assumption in Equation E.1, there are no unary atoms in $root_{\mathcal{T}}(R(x, c))$. Moreover, since $\mathcal{K}, \mathcal{N}$ is a simple evolution setting there are no binary atoms in $root_{\mathcal{T}}(R(x, c))$ besides $R(x, c)$. Thus, $root_{\mathcal{T}}(R(x, c)) = \{R(x, c)\}$. Consider a model

$$\mathcal{J}' = \mathcal{J} \setminus \bigcup_{R(x,c) \in \mathcal{D}} \{R(x, c)\}.$$

We now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Observe that $\mathcal{J}' \models \mathcal{N}$. Assume this is not the case and there is $g \in \mathcal{N}$ such that $g \notin \mathcal{J}'$. Since $\mathcal{J} \models \mathcal{N}$, we conclude that $g \in \bigcup_{R(x,c) \in \mathcal{D}} \{R(x,c)\}$. Therefore, $g = R(x,c)$ for some $R(x,c) \in \mathcal{D}$, we have that $R(x,c) \in \mathcal{N}$. Combining this with the fact that $\{D(c), \exists R^-(c)\} \models_{\mathcal{T}} \bot$ we conclude that $\{D(c)\} \cup \mathcal{N} \models_{\mathcal{T}} \bot$. This contradicts the fact that $\mathcal{N} \not\models_{\mathcal{T}} D(c)$. Thus, $\mathcal{J}' \models \mathcal{N}$. Due to Proposition 16 $\mathcal{J}' \models \mathcal{T}$. Therefore, $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$. By construction of $\mathcal{J}'$, we have that every $R(x,c)$ from $\mathcal{D}$ is in *not* in $\mathcal{I}$ and $\mathcal{J}$, while it is in $\mathcal{J}'$. Therefore, $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$ and we conclude the proof.

Assume $\mathcal{D} = \emptyset$, then consider

$$\mathcal{J}' = \mathcal{J} \cup \mathcal{I}[D(c)].$$

Again, we now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Observe that $\mathcal{J}' \models \mathcal{N}$, since $\mathcal{J}$ does so. Clearly, both $\mathcal{J}$ and $\mathcal{I}[D(c)]$ satisfy $\mathcal{T}$. Due to Proposition 15, to finish the proof of $\mathcal{J}' \models \mathcal{T}$, it remains to show that for every $g_1 \in \mathcal{J}$ and $g_2 \in \mathcal{I}[D(c)]$ it holds that $\{g_1, g_2\} \not\models_{\mathcal{T}} \bot$. Assume this is not the case and there are $g_1 \in \mathcal{J}$ and $g_2 \in \mathcal{I}[D(c)]$ such that $\{g_1, g_2\} \models_{\mathcal{T}} \bot$, then there is an NI α in $cl(\mathcal{T})$ such that $g_1 \to \neg g_2$ instantiates α.

Observe that $g_2 \neq D(c)$. Indeed, if $g_2 = D(c)$, then (since $D(c)$ is a unary atom) α is of the form $D \sqsubseteq \neg B$. If $B = A'$ for some atomic concept A', then $g_1 = A'(c) \in \mathcal{J}$. If $\mathcal{N} \models_{\mathcal{T}} A'(c)$, then $\mathcal{N} \models_{\mathcal{T}} \neg D(c)$, which contradicts the fact that $\mathcal{N} \not\models_{\mathcal{T}} \neg D(c)$. If $\mathcal{N} \not\models_{\mathcal{T}} A'(c)$, then due to Case (i) of Proposition 53 and the facts that $D(c) \in \mathcal{I}$ and $\{D(c), A'(c)\} \models_{\mathcal{T}} \bot$, we conclude that $A'(c) \notin \mathcal{J}$, which gives a contradiction. If $B = \exists R_1$ for some role R_1, then $g_1 = R_1(c,y) \in \mathcal{J}$ for some $y \in \Delta$. Assume that $\mathcal{N} \models_{\mathcal{T}} R_1(c,y)$, then $\mathcal{N} \models_{\mathcal{T}} \neg D(c)$, which gives a contradiction. If $\mathcal{N} \not\models_{\mathcal{T}} R_1(c,y)$, and also $\mathcal{N} \not\models_{\mathcal{T}} \exists R_1(c)$, $\mathcal{N} \not\models_{\mathcal{T}} \exists R_1(y)$, then due to Case (ii) of Proposition 53, and the facts that $D(c) \in \mathcal{I}$ and $\{D(c), R'(c,y)\} \models_{\mathcal{T}} \bot$, we conclude that $R'(c,y) \notin \mathcal{J}$ and again obtain a contradiction. If $\mathcal{N} \not\models_{\mathcal{T}} R_1(c,y)$ and either $\mathcal{N} \models_{\mathcal{T}} \exists R_1(c)$ or $\mathcal{N} \models_{\mathcal{T}} \exists R_1^-(y)$ holds, then, due to $D(c) \in \mathcal{I}$ and $\{D(c), R'(c,y)\} \models_{\mathcal{T}} \bot$, we can apply Proposition 52 and conclude that $R'(c,y) \notin \mathcal{J}'$, thus we obtain a contradiction. Since $\mathcal{K}, \mathcal{N}$ is a simple evolution setting, every $g' \in \mathcal{I}[D(c)]$ is unary and $\mathcal{D}(c) \models_{\mathcal{T}} g'$. Thus, we can apply to such g' the same argument as to $D(c)$ above to obtain a contradiction Thus, due to Proposition 15 $\mathcal{J} \cup \mathcal{I}[D(c)] \models \mathcal{T}$. It remains to show that $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$. This follows from the fact that $\mathcal{I}[D(c)] \subseteq \mathcal{I}$, $\mathcal{I}[D(c)] \subseteq \mathcal{J}'$, while $D(c) \notin \mathcal{J}$.

Case (ii): Assume there is $D(c) \in \mathcal{J}$ and a unary MA $A(c)$ satisfying $\mathcal{K} \models A(c)$, $\mathcal{T} \models A \sqsubseteq D$ and $A(c) \in \mathit{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$, while $A(c) \notin \mathcal{J}$ holds. Let $\mathcal{I}$ be a model of $\mathcal{K}$ such that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Consider

$$\mathcal{J}' = \mathcal{J} \cup \mathcal{I}[A(c)].$$

Again, we now show that $\mathcal{J}' \models \mathcal{T} \cup \mathcal{N}$ and $\mathcal{I} \ominus \mathcal{J}' \subsetneq \mathcal{I} \ominus \mathcal{J}$, which contradicts the fact that $\mathcal{J} \in \mathcal{I} \diamond \mathcal{N}$ under $\mathcal{L}^a_{\subseteq}$. Observe that $\mathcal{J}' \models \mathcal{N}$, since $\mathcal{J}$ does so, and both $\mathcal{J}$ and $\mathcal{I}[A(c)]$ satisfy $\mathcal{T}$. Due to Proposition 15 and Lemma 12 of [5], it suffices to show that for every NI $\alpha \in cl(\mathcal{T})$ and every $g \in \mathcal{J}$ and $f \in \mathcal{I}[A(c)]$, $\{f, g\}$ satisfies α. Assume there is an NI α in $cl(\mathcal{T})$, $g \in \mathcal{J}$ and $f \in \mathcal{I}[A(c)]$ such that $\{g, f\} \models_{\{\alpha\}} \bot$. If $\mathcal{N} \models_{\mathcal{T}} g$, then $\mathcal{N} \cup \{f\} \models_{\{\alpha\}} \bot$. Thus, $A(c) \in \mathit{root}^{\mathcal{I}}_{\mathcal{T}}(f)$ and therefore $A(c) \notin \mathit{Align}_{\mathcal{T}}(\mathcal{I}, \mathcal{N})$ which gives a contradiction. Assume $\mathcal{N} \not\models g$. If g is unary, then we can apply Case (ii) of Proposition 53 to obtain a contradiction. If g is binary, then, as in the proof of Case (i) of the current proposition, we can apply Case (ii) of Proposition 53 or Proposition 52, and obtain a contradiction. □