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Diagonal plane sections of trivariate copulas

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ABSTRACT

We introduce the notion of diagonal plane section of a trivariate copula as an additional tool to describe its tail dependence behavior. This notion extends the concept of diagonal section of a bivariate copula. We provide existence results for trivariate copulas with a given diagonal plane function. Related results are discussed, especially about extensions to a higher dimensional setting.

1. Introduction

The issue of aggregation of different sources of information into a single output has obtained a renovated interest in the last years that has also stimulated a novel treatment of its theoretical foundations. In particular, the formalization of the aggregation processes given by the so-called *aggregation functions* (see [15] for an overview) has underlined properties and relationships among various classes of such functions.

A copula is a special kind of n–ary aggregation function whose input values belong to $\mathbb{I} := [0, 1]$. Copulas are also Lipschitz continuous aggregation functions, i.e. the error in the aggregation output is linearly dependent on the error in the corresponding input terms (see [12,16] for more details). Copulas are also extensively used in statistics in order to aggregate one–dimensional probability laws into a joint (multivariate) probability law, keeping the marginal behavior fixed (see [18,23]). Such a feature is particularly helpful in building (flexible) high–dimensional distribution functions that take into account the available information about the lower–dimensional marginals and the ways they interact.

As stressed in [15], "*the choice of the aggregation function to be used is far from being arbitrary and should be based upon properties dictated by the framework in which the aggregation is performed*". For example, in copula methods, the aim is often to calculate a risk measure associated with a multivariate model, and a possible relevant information that may constrained the choice is connected with the diagonal sections of its bivariate margins, since they describe conveniently the tail behavior of the associated random pairs (see, e.g., [6]). As such, various construction methods have been introduced assuming that the values of the copula are known along diagonals and other suitable sections. See, for instance, [3,4,10,17,19,25,26].

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In this paper, we aim at developing novel insights into this problem. Specifically, in Section 2, we present a possible extension of the concept of diagonal section of a bivariate copula to the three–dimensional case by introducing the so–called *diagonal plane section*. We discuss, hence, existence results for copulas with a given information on a specified subset of diagonal type (Section 3); furthermore, we show how they can be applied into a slightly different setting previously considered in [26] (Section 4). Section 5 is devoted to final comments about possible higher–dimensional generalizations of the introduced concepts.

2. Basic definitions and properties

Let $n \geq 2$ be an integer. An n-dimensional *copula* C (n-*copula*, for short) is an n-ary aggregation function with neutral element 1 that is n-increasing (see, e.g., [15]). Equivalently, C can be seen as a multivariate distribution function whose univariate margins are uniformly distributed on $\mathbb{I}$. Each n-copula C induces an n–fold stochastic measure μ_C on the Borel sets of $\mathbb{I}^n$ by setting $\mu_C(B) = V_C(B)$ for every n–rectangle $B \subseteq \mathbb{I}^n$, where $V_C(B)$ is the C–volume of B. Interestingly, the copula measure is also endowed in the construction of several generalized fuzzy integrals [22]. For more details about copulas, we refer to [12,23].

The *diagonal section* δ_C of an n–copula C is defined as the function given by $\delta_C(t) = C(t, \dots, t)$ for every $t \in \mathbb{I}$. It provides information about the tail concentration (see, e.g., [6]) of the copula C, i.e. the way the probability mass is distributed near the corners $(0, \dots, 0)$ and $(1, \dots, 1)$. In particular, δ_C can be used in order to calculate the tail dependence coefficients associated with C, when they exist (see, e.g., [13,18]).

If C is a 2–copula, then the diagonal section δ_C satisfies: (i) $\delta_C(1) = 1$, (ii) $\delta_C(t) \leq t$ for every $t \in \mathbb{I}$, and (iii) $0 \leq \delta_C(t') - \delta_C(t) \leq 2(t' - t)$ for every $t, t' \in \mathbb{I}$ such that $t \leq t'$. Any function $\delta : \mathbb{I} \to \mathbb{I}$ that satisfies (i), (ii) and (iii), is called a *diagonal function*.

The diagonal section contains the information about the values that an n–copula assumes on a set of (Hausdorff) dimension 1. Here, we provide a possible generalization of this concept when the values of the copula are known on a set of dimension 2. For the sake of simplicity, in the sequel we consider copulas in three dimensions.

Definition 2.1. The (1, 2)–*diagonal plane section* of a 3–copula C is the function $\Delta_C^{1,2} : \mathbb{I}^2 \to \mathbb{I}$ given by

$$\Delta_C^{1,2}(x, y) = C(x, x, y)$$

for every $(x, y) \in \mathbb{I}^2$. Similarly, the (1, 3)–*diagonal plane section* and the (2, 3)–*diagonal plane section* of C are defined by

$$\Delta_C^{1,3}(x, y) = C(x, y, x), \qquad \Delta_C^{2,3}(x, y) = C(y, x, x).$$

If (X, Y, Z) is a random vector (on a suitable probability space) distributed according to the copula C, then $\Delta_C^{1,2}$ represents the distribution function of $(\max(X, Y), Z)$. Similar statements apply to the other diagonal plane sections. As such, any diagonal plane section is a 2–increasing function. Obviously, any diagonal plane section contains the information about the diagonal section of C.

Example 2.1. Let D be the 2–copula that is obtained by spreading the probability mass uniformly on the segments joining (0, 1/2) with (1/2, 1), and (1/2, 1/2) with (1, 0). Consider the 3–copula $C(x, y, z) = zD(x, y)$. The diagonal plane sections associated with C are given by

$$\Delta_C^{1,2}(x, y) = yD(x, x), \quad \Delta_C^{1,3}(x, y) = xD(x, y), \quad \Delta_C^{2,3}(x, y) = xD(y, x).$$

Since D is not exchangeable, the diagonal plane sections are different. In fact, it is enough to consider that

$$\Delta_C^{1,2}(3/4, 1/4) = 1/8, \quad \Delta_C^{1,3}(3/4, 1/4) = 0, \quad \Delta_C^{2,3}(3/4, 1/4) = 3/16.$$

Notice that two bivariate margins of D coincide. Roughly speaking, the information given by the diagonal plane sections is not only related to the bivariate margins of a 3–copula, but also to the way they interact.

In the sequel, we concentrate on the (1, 2)–diagonal plane section, since the other cases can be obtained by a permutation of the arguments of the copula C. For simplicity, we set $\Delta_C := \Delta_C^{1,2}$.

As said, Δ_C has margins equal to the diagonal section of the copula $C_{12}(x, y) := C(x, y, 1)$ and the uniform distribution on $\mathbb{I}$, respectively. Specifically, the following properties are satisfied:

(D1) $\Delta_C(x, 1)$ is a diagonal function;
(D2) $\Delta_C(1, y) = y$ for every $y \in \mathbb{I}$;
(D3) Δ_C is 2–increasing.

Obviously, Δ_C is a 2–dimensional distribution function whose support is contained in $\mathbb{I}^2$. In particular, Sklar's theorem [12] ensures that, if $\delta_{C_{12}}$ denotes the diagonal section of C_{12},

$$\Delta_C(x, y) = D(\delta_{C_{12}}(x), y) \tag{1}$$

for all $(x, y) \in \mathbb{I}^2$ and a suitable 2–copula D.

If the random vector (X, Y, Z) is distributed according to C, Δ_C may also serve to describe the behavior of the conditional distribution function of $[X \leq x, Y \leq x | Z \leq y]$, which represents, roughly speaking, a kind of conditional version of the diagonal section of (X, Y).

Example 2.2. Let C be an Archimedean 3–copula (see, for instance, [1]) with additive generator φ. Then

$$\Delta_C(x, y) = \varphi^{-1}(2\varphi(x) + \varphi(y)).$$

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Since $\Delta_C(x,y)$ is a continuous distribution function, in view of Sklar's Theorem [12], it is associated with a 2–copula that, in such a case, coincides with the Archimedean 2–copula generated by φ, i.e. it coincides with the bivariate margin C_{12}.

Example 2.3. Let C be the 3–copula defined by

$$C(x,y,z) = x_{(1)} f(x_{(2)}) f(x_{(3)}),$$

where $x_{(1)} \le x_{(2)} \le x_{(3)}$ are the order statistics of (x,y,z), and f is a suitable generator as defined in [9,11]. It easily follows that

$$\Delta_C(x,y) = \min(x,y) f(x) f(\max(x,y))$$

and its marginal $\Delta_C(x,1) = x f(x)$. In particular, if $f(t) = t^\alpha$ for $\alpha \in \mathbb{I}$, then the copula of Δ_C is given by

$$D(u,v) = \Delta_C(u^{1/(\alpha+1)}, v) = \begin{cases} uv^\alpha, & u^{1/(\alpha+1)} < v, \\ u^{2\alpha/(\alpha+1)} v, & u^{1/(\alpha+1)} > v. \end{cases}$$

Such a D is, in general, non–exchangeable and contains a singular component along the set $\{(u,v) \in \mathbb{I}^2 : u^{1/(\alpha+1)} = v\}$. In particular, D does not coincide with any bivariate margin of C

These latter examples show that there is no direct relationship between D and (the bivariate margins of) C.

3. Existence of 3–copulas with a given diagonal plane section

In the bivariate case, any diagonal function is the diagonal section of (at least) a copula. An example is given by the copula

$$K_\delta(x,y) = \min\left(x, y, \frac{\delta(x)+\delta(y)}{2}\right), \tag{2}$$

see [14,20,24].

Analogously, we may wonder whether the information about the diagonal plane section allows us to construct (at least) a 3–copula that is compatible with it. To this end, we call a function $\Delta : \mathbb{I}^2 \to \mathbb{I}$ that satisfies properties (D1)–(D3) a *diagonal plane function*. Interestingly, any diagonal plane function Δ is also the diagonal plane section of a trivariate copula C. The proof of this result is grounded on measure-theoretic arguments and, in particular, on the following version of the Disintegration Theorem (see, e.g., [2]).

Lemma 3.1. *Let μ be a probability measure on $\mathbb{I}^n$. Let $\pi : \mathbb{I}^n \longrightarrow \mathbb{I}$ be a Borel function and let ν be the push-forward of μ under π, i.e. $\nu(B) := \mu(\pi^{-1}(B))$ for any Borel set $B \subseteq \mathbb{I}$. Then there exists a ν-a.e. uniquely determined Borel family of probability measures $\{\mu_x\}_{x\in\mathbb{I}}$ on the Borel sets of $\mathbb{I}^n$ such that*

$$\mu_x\big(\mathbb{I}^n \setminus \pi^{-1}(x)\big) = 0, \quad \text{for } \nu\text{-a.e. } x \in \mathbb{I}$$

and

$$\int_{\mathbb{I}^n} f(\mathbf{x})\, d\mu(\mathbf{x}) = \int_{\mathbb{I}} \left(\int_{\pi^{-1}(x)} f(\mathbf{x})\, d\mu_x(\mathbf{x}) \right) d\nu(x),$$

for every Borel map $f : \mathbb{I}^n \longrightarrow [0,+\infty]$.

Theorem 3.2. *Let Δ be a diagonal plane function. Then there exists a 3-copula C whose diagonal plane section is Δ.*

Proof. Since Δ is a distribution function, let denote by μ_Δ its associated probability measure. Let $\delta(u) := \Delta(u,1)$ for every $u \in \mathbb{I}$. In view of Lemma 3.1 for $\pi(x,y) = x$, there exists a family of measures $\{\mu_{\Delta,u}\}_{u\in\mathbb{I}}$ in the Borel sets of $\mathbb{I}$ such that

$$\int_{\mathbb{I}^2} f(u,v)\, d\mu_\Delta(u,v) = \int_{\mathbb{I}} \left(\int_{\mathbb{I}} f(u,v)\, d\mu_{\Delta,u}(v) \right) \delta'(u)\, du, \tag{3}$$

for every Borel map $f : \mathbb{I}^2 \longrightarrow [0,+\infty]$. Now, let C_δ be a 2-copula with diagonal section δ whose existence is guaranteed by the fact that δ is a diagonal function. We define the function C in the following manner:

$$C(x,y,z) = \int_0^x \int_0^y \mu_{\Delta,\max(s,t)}([0,z]) d\mu_{C_\delta}(s,t). \tag{4}$$

Note that, for $a \in \mathbb{I}$, we have $\mu_{\Delta,\max(s,t)}([0,z]) = \mu_{\Delta,a}([0,z])$ for every (s,t) belonging to the union of the two segments $[0,a] \times \{a\}$ and $\{a\} \times [0,a]$. By using (3) with $f(u,v) = \mathbf{1}_{[0,x]\times[0,y]}(u,v)$, we obtain

$$\begin{aligned} \Delta_C(x,z) = C(x,x,z) &= \int_0^x \int_0^x \mu_{\Delta,\max(s,t)}([0,z])\, d\mu_{C_\delta}(s,t) \\ &= \int_0^x \mu_{\Delta,\xi}([0,z]) \delta'(\xi)\, d\xi \\ &= \int_0^x \left(\int_0^z d\mu_{\Delta,\xi}(\eta) \right) \delta'(\xi)\, d\xi = \Delta(x,z). \end{aligned}$$

Moreover, since C_δ is a 2-copula, we have

$$C(x,1,1)=\int_0^x\int_0^1 \mu_{\Delta,\max(s,t)}(\mathbb{I})\,d\mu_{C_\delta}(s,t)=\int_0^x\int_0^1 1\,d\mu_{C_\delta}(s,t)=x$$

for every $x \in \mathbb{I}$, and, similarly, $C(1,y,1)=y$ for every $y \in \mathbb{I}$. Moreover, $C(1,1,z)=z$ for every $z \in \mathbb{I}$, since $C(1,1,z)=\Delta(1,z)=z$. Thus, all the univariate margins of C are uniform on $\mathbb{I}$.

Finally, from its definition, it is straightforward to check that C is the restriction to $\mathbb{I}^3$ of a distribution function and, thus, is 3-increasing. We conclude that C is a 3-copula. □

In the next result we study under which conditions on Δ the 3-copula C of Theorem 3.2 is unique.

Proposition 3.3. *Let Δ be a diagonal plane function. Then there exists a unique 3-copula C whose diagonal plane section is Δ if, and only if, $\Delta(x,1)=x$ for every $x \in \mathbb{I}$.*

Proof. Suppose there exists $x \in \mathbb{I}$ such that $\Delta(x,1)<x$, and let $\delta(x)=\Delta(x,1)$ for every $x \in \mathbb{I}$. Then there are infinitely many 2–copulas that have δ as their diagonal section (see, e.g., [5]) and, hence, in view of (4) they generate different 3-copulas with the same diagonal plane section Δ.

On the other hand, if $\Delta(x,1)=x$ for all $x \in \mathbb{I}$, then any 3-copula whose diagonal plane section is equal to Δ must have its mass concentrated on the surface $\{(x,y,z) \in \mathbb{I}^3 : x=y\}$. Thus, its (unique) expression derived from (4) is given by $C(x,y,z)=\Delta(\min(x,y),z)$. □

We now study the existence of 3-copulas with a prescribed diagonal plane section such that their induced measure is singular or absolutely continuous (for these concepts, see for instance [12]). To this end, for any integer $n \geq 1$, we denote by λ_n the n–dimensional Lebesgue measure.

Proposition 3.4. *Let Δ be a diagonal plane function. Then there exists a singular 3-copula C whose diagonal plane section is Δ.*

Proof. Let δ be the diagonal function given by $\delta(t):=\Delta(t,1)$. Consider the copula K_δ given by (2). The probability mass of C_δ is concentrated on a hairpin-like set A which is formed by the graphs of an increasing function $f:\mathbb{I}\to\mathbb{I}$ and its symmetrical with respect to the main diagonal of the unit square (see, e.g., [7]). In view of the proof of Theorem 3.2, the 3-copula C given by (4) has its mass concentrated on the set $E=A\times\mathbb{I}$. Therefore, $\lambda_2(E)=0$, and hence C is a singular copula. □

Proposition 3.5. *Let Δ be a diagonal plane function. Then there exists an absolutely continuous 3-copula C whose diagonal plane section is Δ if, and only if, Δ is absolutely continuous and the set $J=\{t\in\mathbb{I}:\Delta(t,1)\neq t\}$ has Lebesgue measure 1.*

Proof. Let $\delta(t)=\Delta(t,1)$ for every $t \in \mathbb{I}$.

Suppose there is an absolutely continuous 3-copula C with $\Delta_C=\Delta$. Then $C_3(x,y)=C(x,y,1)$ is an absolutely continuous 2-copula with diagonal section δ, which implies that the set J has Lebesgue measure 1 as a consequence of [8, Theorem 3.1]. Moreover, Δ_C is also absolutely continuous and its density $f(x,y)$ is given by

$$\int_0^x (c(x,t,y)+c(t,x,y))dt,$$

where c is the density of C.

On the other hand, if the set J has Lebesgue measure 1, then [8, Theorem 3.1] implies that there exists an absolutely continuous 2–copula C_δ with density c_δ and diagonal section δ. Now, consider the copula C of (4), which can be written as

$$\begin{aligned}C(x,y,z)&=\int_0^x\int_0^y \mu_{\Delta,\max(s,t)}([0,z])\,d\mu_{C_\delta}(s,t)\\&=\int_0^x\int_0^y\int_0^z f_{\max(s,t)}(z)c_\delta(s,t)\,du\,ds\,dt,\end{aligned}$$

where f_ξ is the density of $\mu_{\Delta,\xi}$, which is absolutely continuous (with respect to λ_1) since Δ is absolutely continuous. Therefore, C is absolutely continuous. □

4. Extension to copulas with a given sub–diagonal plane section

The study of diagonal section of a 2–copula has been generalized to sub–diagonal sections in [26]. We recall basic notions and properties from this latter reference.

Given a 2–copula C and $x_0 \in]0,1[$, the *sub–diagonal section* δ_{C,x_0} of C at x_0 is the function $\delta_{C,x_0}:[0,1-x_0]\to[0,1-x_0]$ defined by $\delta_{C,x_0}(t)=C(x_0+t,t)$. If (X,Y) is a random pair distributed according to the 2-copula C, δ_{C,x_0} is the restriction to $[0,1-x_0]$ of the distribution function of $\max(X-x_0,Y)$. Notice that a similar concept of super–diagonal section can be defined as in [26].

Given $x_0 \in]0,1[$, a *sub–diagonal function* δ_{x_0} is a function from $[0,1-x_0]$ into $[0,1-x_0]$ with the following properties:

(i) $\delta_{x_0}(1-x_0)=1-x_0$,
(ii) $\delta_{x_0}(t)\leq t$ for every $t\in[0,1-x_0]$,
(iii) $0\leq\delta_{x_0}(t')-\delta_{x_0}(t)\leq 2(t'-t)$ for every $t,t'\in[0,1-x_0]$ with $t\leq t'$.

From [26, Corollary 3] it is known that, given a sub-diagonal function δ_{x_0} there are infinitely many copulas C whose sub-diagonal section at x_0 coincides with δ_{x_0}.

As above, we can now extend the concept of sub-diagonal section.

Definition 4.1. For any 3-copula C and $x_0 \in]0, 1[$, we define the $(1, 2)$–*sub–diagonal plane section* of C at x_0 as the function $\Delta_{C,x_0} : [0, 1 - x_0] \times \mathbb{I} \to \mathbb{I}$ defined by $\Delta^{1,2}_{C,x_0}(x, y) = C(x_0 + x, x, y)$ for every $(x, y) \in [0, 1 - x_0] \times \mathbb{I}$.

Similarly, the $(1, 3)$–*sub–diagonal plane section* and the $(2, 3)$–*sub–diagonal plane section* of C can be defined as well. Here we consider $\Delta^{1,2}_{C,x_0}$ since the other cases can be treated analogously. For the sake of simplicity, in the sequel we write Δ_{C,x_0} for the (1, 2)–sub–diagonal plane section of C at x_0.

For any 3–copula C and $x_0 \in]0, 1[$, Δ_{C,x_0} is the measure–generating function of the measure $\widetilde{\mu}$ defined on any rectangle $B = [a_1, b_1] \times [a_2, b_2] \subseteq [0, 1 - x_0] \times \mathbb{I}$ by

$$\widetilde{\mu}(B) = V_C([a_1, b_1 + x_0] \times [a_1, b_1] \times [a_2, b_2]).$$

In fact, for $B = [0, x] \times [0, y]$, $\widetilde{\mu}([0, x] \times [0, y]) = \Delta_{C,x_0}(x, y)$. The measure $\widetilde{\mu}$ can be hence extended by standard arguments to all Borel sets of $[0, 1 - x_0] \times \mathbb{I}$. Its total mass is given by $1 - x_0$.

The marginal measure-generating functions associated with Δ_{C,x_0} are given by

$$\Delta_{C,x_0}(x, 1) = \delta_{C_{12},x_0},$$

i.e. the sub–diagonal section at x_0 of the copula $C_{12}(x, y) = C(x, y, 1)$, and

$$\Delta_{C,x_0}(1 - x_0, y) = C_{23}(1 - x_0, y),$$

i.e. the horizontal section at $(1 - x_0)$ of the copula $C_{23}(x, y) = C(1, x, y)$ (for this concept see [21]). Mimicking the case of diagonal plane section, we can therefore give the following definition.

Definition 4.2. A function $\Delta_{x_0} : [0, 1 - x_0] \times \mathbb{I} \longrightarrow [0, 1 - x_0]$ is called a *sub–diagonal plane function* at $x_0 \in]0, 1[$ if it satisfies the following properties:

(S1) $\Delta_{x_0}(x, 1)$ is a sub–diagonal function;
(S2) $\Delta_{x_0}(x, y)$ is 1–Lipschitz in y for every $x \in [0, 1 - x_0]$;
(S3) Δ_{x_0} is the measure–generating function of a measure μ defined on the Borel sets of $[0, 1 - x_0] \times \mathbb{I}$.

As it was done before, given a sub–diagonal plane function Δ_{x_0} one may wonder whether a 3-copula C exists such that $\Delta_{C,x_0} = \Delta_{x_0}$. The answer is positive as stated below.

Theorem 4.1. *Let Δ_{x_0} be a sub–diagonal plane function at $x_0 \in]0, 1[$. Then there exists a 3-copula C whose sub–diagonal plane section at x_0 is Δ_{x_0}.*

Proof. Let $\delta_{x_0}(x) = \Delta_{x_0}(x, 1)$ for every $x \in [0, 1 - x_0]$. Let $\mu_{\Delta_{x_0}}$ be the measure associated with Δ_{x_0}. From Lemma 3.1, there exists a family of measures $\{\mu_{\Delta_{x_0},u}\}_{u\in[0,1-x_0]}$ such that

$$\int_0^{1-x_0}\int_0^1 f(u, v)d\mu_{\Delta_{x_0}}(u, v) = \int_0^{1-x_0}\left(\int_0^1 f(u, v)d\mu_{\Delta_{x_0},u}(v)\right)\delta'_{x_0}(u)du$$

for every Borel map $f : [0, 1 - x_0] \times \mathbb{I} \to \mathbb{R}$.

Let C_δ be a 2–copula with sub–diagonal section at x_0 equal to δ_{x_0}, whose existence is guaranteed from [26, Corollary 3]. Consider the function $a(z) = z - \Delta_{x_0}(1 - x_0, z)$ for $z \in \mathbb{I}$. As a consequence of (S1) and (S2), a is increasing and $a(1) = x_0$. Let b be the distribution function $b(z) = a(z)/x_0$.

Define the function

$$\begin{aligned} C(x, y, z) = &\int_0^1\int_0^{1-x_0} \chi_{[0,x]\times[0,y]}(s, t)\mu_{\Delta_{x_0},\max(s-x_0,t)}([0, z])\, d\mu_{C_\delta}(s, t) \\ &+ \int_0^1\int_{1-x_0}^1 \chi_{[0,x]\times[0,y]}(s, t)b(z)\, d\mu_{C_\delta}(s, t), \end{aligned}$$

where χ_A is the indicator function of the set A.

For $z = 1$, $C(x, y, 1) = C_\delta(x, y)$ for every $(x, y) \in \mathbb{I}^2$. Thus C has uniform margins in the first and second coordinate. Moreover, for every $z \in \mathbb{I}$,

$$C(1, 1, z) = \Delta_{x_0}(1 - x_0, z) + b(z) \cdot x_0 = z.$$

Since, by definition, C is 3–increasing, it follows that it is a copula.

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Finally, it remains to prove that $C(x_0 + x, x, y) = \Delta_{x_0}(x, y)$ for every $(x, y) \in [0, 1 - x_0] \times \mathbb{I}$. In fact,

$$\int_0^1 \int_0^{1-x_0} \chi_{[0,x]\times[0,y]}(s,t)\mu_{\Delta_{x_0},\max(s-x_0,t)}([0,z])\, d\mu_{C_\delta}(s,t)$$
$$= \int_0^1 \int_0^{1-x_0} \chi_{[0,x]\times[0,y]}(s,t)\mu_{\Delta_{x_0},t}([0,z])\delta'_{x_0}(t)dt = \Delta_{x_0}(x,y),$$

which concludes the proof. □

Now, by using the copulas introduced in [26] and the findings in [4], the following results can be proved by using similar arguments as in previous section.

Corollary 4.2. *Let Δ_{x_0} be a sub–diagonal plane function at $x_0 \in]0, 1[$. Then there exist infinitely many 3–copulas whose sub–diagonal plane section at x_0 is Δ_{x_0}.*

Corollary 4.3. *Let Δ_{x_0} be a sub–diagonal plane function at $x_0 \in]0, 1[$. Then there exists a singular 3–copula C whose sub–diagonal plane section at x_0 is Δ_{x_0}.*

Corollary 4.4. *Let Δ_{x_0} be an absolutely continuous sub–diagonal plane function at $x_0 \in]0, 1[$. Then there exists an absolutely continuous 3-copula C whose sub–diagonal plane section at x_0 is Δ_{x_0}.*

5. Conclusions

The concept of diagonal plane section of a 3–copula has been introduced in order to generalize the notion of diagonal section of a 2–copula. Several properties and existence results have been discussed. The presentation has been focused on 3–copulas, but higher dimensional extensions could be considered as well. In fact, from our viewpoint, a general notion of diagonal hyper-plane section can be given as follows.

For any integer $n \geq 3$, let C be an n-copula. Let $1 \leq k \leq n$. We can define the $(1, \ldots, k)$–*diagonal hyper-plane section* of C as the function $\Delta_C^{1,\ldots,k} : \mathbb{I}^{n-k+1} \to \mathbb{I}$ given by

$$\Delta_C^{1,\ldots,k}(x_1, \ldots, x_{n-k+1}) = C(\mathbf{y}),$$

where $\mathbf{y} = (y_1, \ldots, y_n)$ is the vector given by $y_1 = \cdots = y_k = x_1$, $y_{k+j-1} = x_j$ for $j = 2, \ldots, n-k+1$. In other words, $\Delta_C^{1,\ldots,k}$ is calculated from C by assuming that its first k input values are all equal. In particular, if $k = n$, we obtain the usual notion of diagonal section of C. Obviously, if $k = 1$, $\Delta_C^1 = C$.

If $(X_1, \ldots, X_n)$ is a random vector distributed according to C, then $\Delta_C^{1,\ldots,k}$ is the distribution function of $(\max_{i=1,\ldots,k} X_i, X_{k+1}, \ldots, X_n)$. In other words, $\Delta_C^{1,\ldots,k}$ contains the information about the dependence between the last $(n-k)$ components of a random vector and a suitable aggregation of the first k ones.

In general, if we consider $(i_1, \ldots, i_k)$ a sub-vector of indices of $\{1, 2, \ldots, n\}$, $i_1 < i_2 < \cdots < i_k$, we can define the $(i_1, \ldots, i_k)$–*diagonal hyper-plane section* of C as the function $\Delta_C^{i_1,\ldots,i_k} : \mathbb{I}^{n-k+1} \to \mathbb{I}$ given by

$$\Delta_C^{i_1,\ldots,i_k}(x_1, \ldots, x_{n-k+1}) = C(\mathbf{y}^\sigma),$$

where $\mathbf{y}^\sigma$ is the vector derived from $\mathbf{y}$ by means of a permutation of its components given by $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ where $\sigma(j) = i_j$ for $j = 1, \ldots, k$, while the other arguments are held fixed.

The determination of high–dimensional copulas with given diagonal hyper-plane functions remains an open problem, since a direct extension of the previous arguments (mainly based on the disintegration of a measure) does not seem to be applicable.

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